Math 2300-007: Quiz 10

Name: Solutions 4/13/18 Score: ________

Collaborators:

Directions: This take-home quiz will be due at the beginning of class on Tuesday, April 10. You may use your notes, textbook, and colleagues from our class as resources, but your final write-up should be in your own words. If you work with collaborators from our class, please include their names on this quiz.

1. To what value does the series $\sum_{n=1}^{\infty} \frac{(-9)^n}{(2n)!}$ converge?

The goal is to figure out which Taylor series something was plugged into to obtain $\sum_{n=1}^{\infty} \frac{(-9)^n}{(2n)!}$. The $(2n)!$ in the denominator hints that maybe $\cos(x)$ is a good place to start...

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos(3) = 1 - \frac{3^2}{2!} + \frac{3^4}{4!} - \frac{3^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n}}{(2n)!}$$

Now, $3^{2n} = (3^2)^n = 9^n$, so we have

$$\cos(3) = \sum_{n=0}^{\infty} \frac{(-1)^n 9^n}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-9)^n}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-9)^n}{(2n)!}$$

The mystery series starts at $n=1$, so write the 0^{th} term separately

Hence,

$$\boxed{\cos(3) - 1 = \sum_{n=1}^{\infty} \frac{(-9)^n}{(2n)!}}$$

2. Find $T_4(x)$, the fourth degree Taylor Polynomial for $f(x) = \sqrt{x}$ at $a = 1$. If you use $T_4(2)$ as an estimate for $\sqrt{2}$, what does Taylor's Inequality say about the error in your estimate?

	at $x=1$
$f(x) = \sqrt{x} = x^{1/2}$	$f(1) = 1$
$f'(x) = \frac{1}{2}x^{-1/2}$	$f'(1) = \frac{1}{2}$
$f''(x) = \frac{-1}{4}x^{-3/2}$	$f''(1) = -\frac{1}{4}$
$f'''(x) = \frac{3}{8}x^{-5/2}$	$f'''(1) = \frac{3}{8}$
$f^{(4)}(x) = \frac{-15}{16}x^{-7/2}$	$f^{(4)}(1) = \frac{-15}{16}$
$f^{(5)}(x) = \frac{105}{32}x^{-9/2}$	use this one for Error Part

We have

$$T_4(x) = 1 + \frac{1}{2}(x-1) - \frac{\frac{1}{4}}{2!}(x-1)^2 + \frac{\frac{3}{8}}{3!}(x-1)^3 - \frac{\frac{15}{16}}{4!}(x-1)^4$$

$$T_4(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$$

An estimate for $\sqrt{2}$ is

$$\sqrt{2} \approx T_4(2) = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} = \frac{179}{128}$$

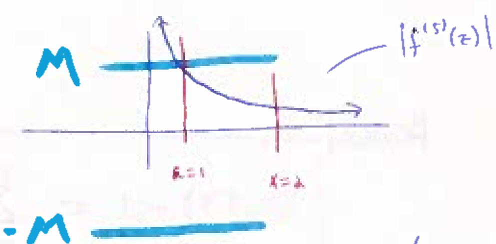
$$\approx 1.398$$

By Taylor's Inequality, the error $|R_4(2)|$ is bounded as follows:

$$|R_4(2)| \leq \frac{M}{(4+1)!}|2-1|^{4+1}$$

, where M is a y-value larger than $|f^{(4+1)}(z)|$ between $a=1$ and $x=2$. We know

$$|f^{(5)}(z)| = \left|\frac{105}{32\sqrt{x^9}}\right|$$



Choosing any M larger than (or equal to) $\frac{105}{32}$ is an upper bound for $|f^{(5)}(z)|$.

$$\leq \frac{105/32}{5!} \cdot 1^5$$

$$= \frac{7}{256}$$

$$\approx 0.027.$$

This says the most we could be off when we use $T_4(2) = \frac{179}{128}$ in place of $\sqrt{2}$ is $0.027\ldots$

3. Why does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge to e^x for all x? Why is it not enough to just use the ratio test to find the interval of convergence of the series?

We can't just use the ratio test because the ratio test would give us an interval of convergence where $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges, but it wouldn't tell us <u>what</u> $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to. Saying that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges <u>to</u> e^x is more specific than saying $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges.

In order to show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x, we will show that the remainder (error) $|R_n(x)|$ approaches 0 for large n. In other words, we will show that $\lim_{n\to\infty} |R_n(x)| = 0$ for any x in $\mathbb{R}$.

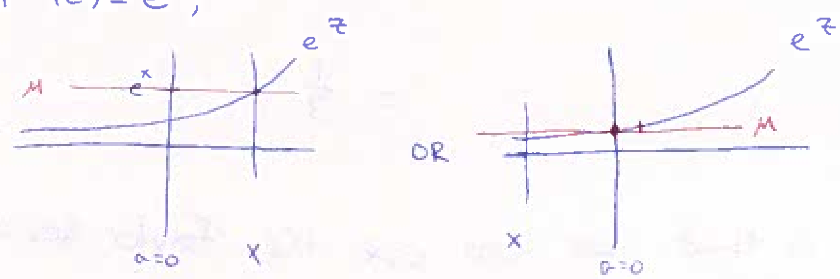
By Taylor's Inequality, we know that for a fixed x,

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

[$a=0$ because the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is centered at $a=0$]

where $M \ge |f^{(n+1)}(z)|$ between $a=0$ and x.

Now, $f^{(n+1)}(z) = e^z$,



Case 1: $x \ge 0$, $M = e^x$

Case 2: $x < 0$, $M = 1$

Factorials bigger than exponential

We have:

$$|R_n(x)| \le \frac{\max(e^x,1)}{(n+1)!} \cdot |x-0|^{n+1} = \max(e^x,1) \cdot \frac{x^{n+1}}{(n+1)!}.$$

$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$, so $\lim_{n\to\infty} \max(e^x,1)\frac{x^{n+1}}{(n+1)!} = 0$. It follows by the squeeze Theorem that $|R_n(x)| \xrightarrow[n\to\infty]{} 0$. We conclude that $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for all x.

4. Use Taylor Series to find $\lim_{x\to 0} \frac{x - \arctan(x)}{x^3}$.

Arctan(x) is the complicated part. We can replace it with its Taylor Series:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \text{ for } x \text{ in } (-1,1)$$

Consequently,

$$\lim_{x\to 0} \frac{x - \arctan(x)}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right)}{x^3}$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \cancel{x} + \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots}{x^3}$$

$$= \lim_{x\to 0} \frac{1}{3} - \frac{x^2}{5} + \frac{x^4}{7} - \cdots$$

$$= \frac{1}{3}.$$

The point is that we can use the Taylor Series $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ in place of arctan(x). It is much easier to do math with polynomials than functions like arctan(x).