Goal: An introduction to the idea of recursively defined sequences, meaning: sequences where each term is defined by a formula involving the previous term (or terms).

1. Sometimes sequences can be described recursively in addition to their more familiar explicit forms.

   (a) Consider the sequence defined by $a_1 = 5, \quad a_n = a_{n-1} + 3$. Write down the first 5 terms of the sequence. Identify what type of sequence it is, and find an explicit formula for a_n.

   (b) Now look at the sequence defined by $a_1 = 3, \quad a_n = 2a_{n-1}$. Write down the first 5 terms of the sequence. Identify what type of sequence it is, and find an explicit formula for a_n.

   (c) Now look at the sequence defined by $a_1 = 5, \quad a_n = na_{n-1}$. Write down the first 5 terms of the sequence. Identify what type of sequence it is, and find an explicit formula for a_n.

2. A sequence that is increasing and bounded above must converge. A sequence that is decreasing and bounded below also must converge. Draw a picture and explain intuitively why this must be so.

3. Suppose that the sequence a_n converges and that $\lim_{n\to\infty} a_n = L$. Does the sequence a_{n+1} converge, and if so what is $\lim_{n\to\infty} a_{n+1}$? Explain.

4. Consider the recursive sequence defined by $a_1 = \sqrt{2}, \quad a_{n+1} = \sqrt{2 + a_n}$

   (a) Write out the first 4 terms, and calculate their approximate values.

   (b) Every term in this sequence is bounded above by 2. For example, let's look at a_4. Explain why $a_4 = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} < \sqrt{2 + \sqrt{2 + \sqrt{2 + 2}}}$, then simplify to show $a_4 < 2$.

   (c) Now show that a_n is increasing by showing that $a_{n+1} > a_n$. You'll need to use the fact that $2 > a_n$.

   $$a_{n+1} = \sqrt{2 + a_n} >$$

   (d) Explain why a_n must converge.

(e) Now we will figure out what it converges to. Let's give a name to its limit, $\lim_{n\to\infty} a_n = L$. Take the limit of both sides of the equation below, and solve for L.

$$a_{n+1} = \sqrt{2 + a_n}$$

5. Here's another recursively defined sequence $\{F_n\}$, called the sequence of *Fibonacci numbers* (which are purported to show up in nature, science, art, architecture, etc.):

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = F_1 + F_2 = 1 + 1 = 2, \quad F_4 = F_2 + F_3 = 1 + 2 = 3, \ldots,$$
$$F_n = F_{n-2} + F_{n-1}.$$

That is, the first two terms are by definition set equal to 1, and each subsequent term is the sum of the previous two.

(a) Compute and write down, F_5, F_6, F_7, F_8, F_9, and F_{10}.

(b) Does $\lim_{n\to\infty} F_n$ look like it exists? If so, what do you think this limit is? If not, why not?

(c) Now let's look at the sequence $\{R_n\}$ of *ratios* of successive Fibonacci numbers:

$$R_1 = \frac{F_2}{F_1} = \frac{1}{1} = 1, \quad R_2 = \frac{F_3}{F_2} = \frac{2}{1} = 2, \quad R_3 = \frac{F_4}{F_3} = \frac{3}{2} = 1.5,$$
$$R_4 = \frac{F_5}{F_4} = \frac{5}{3} = 1.666\ldots, \quad R_n = \frac{F_{n+1}}{F_n}.$$

Find the approximate value of R_5, R_6, R_7, R_8, and R_9.

(d) Does $\lim\limits_{n\to\infty} R_n$ look like it exists? If so, what do you think this limit is? If not, why not?

(e) Let's *assume* for now that $\lim\limits_{n\to\infty} R_n$ exists and is non-zero: let's denote this limit by L. We're going to sneakily compute L. Here's how: start with the equation

$$F_n = F_{n-2} + F_{n-1}$$

defining the Fibonacci numbers. Divide both sides by F_{n-1}. This should give you an equation relating R_{n-1} and R_{n-2}. Write down a simplified version of this equation.

(f) Now, take the limit of both sides of your above equation. What equation do you get, in terms of the limit L? (Hint: if the R_n's have a limit, then whatever they tend to as $n \to \infty$, R_{n-1} and R_{n-2} should tend to the same thing.)

(g) Now, *solve* your above equation for L. (Some hints: (a) you may want to first do some algebra, and then apply the quadratic formula. (b) This will give you two possible solutions; why can you disregard one of them?)

(h) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_n}$ converge or diverge?