MATH 2300 – review problems for Exam 1 ANSWERS

1. Evaluate the integral $\int \sin x \cos x \, dx$ in each of the following ways: This one is self-explanatory; we leave it to you.

   (a) Integrate by parts, with $u = \sin x$ and $dv = \cos x \, dx$. The integral you get on the right should look much like the one you started with, so you can solve for this integral. (Some people call this the "boomerang" method.)

   (b) Integrate by parts, with $u = \cos x$ and $dv = \sin x \, dx$.

   (c) Substitute $w = \sin x$.

   (d) Substitute $w = \cos x$.

   (e) First use the fact that $\sin x \cos x = \frac{1}{2}\sin(2x)$, and then antidifferentiate directly.

   (f) Show that answers to parts (a)–(e) of this problem are all the same. It may help to use the identities $\cos^2 x + \sin^2 x = 1$ and $\cos(2x) = 1 - 2\sin^2 x$.

2. Let $f(x)$ be a continuous function on the set of all real numbers. Show that

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(x)\, dx.$$

   We put $u = e^x$, so that $du = e^x \, dx$. Also, when $x = 0$, $u = e^0 = 1$; when $x = 1$, $u = e^1 = e$. So

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(u)\, du,$$

   which is the same as $\int_1^e f(x)\, dx$.

3. (a) Explain why the integral

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2-4}}$$

   is improper. The integrand becomes infinite at $x = 2$.

   (b) Show that

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2-4}} = \sqrt{21}.$$

   We put $u = x^2 - 4$, to get

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2-4}} = \frac{1}{2}\int_0^{21} \frac{du}{\sqrt{u}} = \frac{1}{2}\lim_{a\to 0^+}\int_a^{21} \frac{du}{\sqrt{u}} = \lim_{a\to 0^+}\left((21)^{1/2} - a^{1/2}\right) = \sqrt{21}.$$

4. Suppose that $\int_0^1 f(t)dt = 5$. Calculate the following:

   (a) $\int_0^{0.5} f(2t)dt$ 5/2

   (b) $\int_0^1 f(1-t)dt$ 5

   (c) $\int_1^{1.5} f(3-2t)dt$ 5/2.

5. Evaluate the following integrals:

(a) $\int 2x\cos(x^2)\,dx$ $\sin\left(x^2\right)+C$

(b) $\int e^{2x}\sin(2x)\,dx$ $\frac{1}{4}e^{2x}(\sin(2x)-\cos(2x))+C$

(c) $\int \cos^2\theta\,d\theta$ $\frac{\theta}{2}+\frac{1}{4}\sin(2\theta)+C=\frac{\cos(\theta)\sin(\theta)+\theta}{2}+C$

(d) $\int x^2\sin(x)dx$ $-x^2\cos x+2x\sin x+2\cos x+C$

(e) $\int \frac{1}{x^2\sqrt{16-x^2}}\,dx = -\frac{\sqrt{16-x^2}}{16x}+C$

(f) $\int \frac{3x^2+6}{x^2(x^2+3)}\,dx$ $-\frac{2}{x}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+C$ (The partial fractions decomposition is $\frac{2}{x^2}+\frac{1}{x^2+3}$.)

(g) $\int \sqrt{25-x^2}\,dx$ $\frac{1}{2}\left(x\sqrt{25-x^2}+25\sin^{-1}\left(\frac{x}{5}\right)\right)+C$

(h) $\int \frac{3x-1}{x^2-5x+6}\,dx$ $8\ln|x-3|-5\ln|x-2|+C$

(i) $\int \sin^3(5x)\cos(5x)\,dx$ $\frac{1}{20}\sin^4(5x)+C$

(j) $\int_2^3 \frac{x^2}{1+x^3}\,dx$ $\frac{\ln(28)-\ln(9)}{3}=\frac{1}{3}\ln\left(\frac{28}{9}\right)$

(k) $\int_0^3 x\,e^{x^2}\,dx$ $\frac{1}{2}\left(e^9-1\right)$

(l) $\int x^7e^{x^4}\,dx$ $\frac{1}{4}e^{x^4}\left(x^4-1\right)+C$

(m) $\int (\ln(x))^2\,dx$ $2x+x\ln^2(x)-2x\ln(x)+C$

6. Evaluate the following integrals

   (a) $\int y\sqrt{y^2+1}\,dy$ Use $w=y^2+1$ to get $\frac{1}{3}\left(y^2+1\right)^{3/2}+C$

   (b) $\int y\sqrt{y+1}\,dy$ Use $w=y+1$ to get $\frac{2(y+1)^{\frac{5}{2}}}{5}-\frac{2(y+1)^{\frac{3}{2}}}{3}+C$

7. (a) Calculate $\int_2^4 \frac{dx}{(x-3)^2}$, if it exists. does not converge

   (b) Find $\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$, if it converges. $\frac{\pi}{2}$ (the integral turns into $\int_0^{\infty}\frac{du}{u^2+1}$.)

8. Estimate $\int_1^2 \ln x\,dx$, by subdividing the interval $[1,2]$ into eight equal parts, and using:

   (a) A left-hand Riemann sum: 0.342322

   (b) A right-hand Riemann sum: 0.428965

   (c) The midpoint rule: 0.386619

   (d) The trapezoid rule: 0.385643

(e) Simpson's rule: 0.386294

(f) Get a bound on the error for the above approximations using midpoint rule and trapezoid rule. $f''(x) = -x^{-2}$ is decreasing, so on our interval $|f''(x)|$ is is largest at $x = 1$. This gives $K = f''(1) = 1$. So $|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{1}{384} < .0027$. And $|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{1}{192} < .006$.

(g) Evaluate the integral more exactly using technology, and comment on how your estimates compare to it.Wolfram alpha gives 0.38629. My curve is increasing and concave down, so as expected my left-hand estimate is too low, my right-hand estimate is too large, by trapezoidal estimate is too small, and my midpoint estimate is too large. As expected, the true value liest between the trapezoidal estimate and the midpoint estimate. My error with the midpoint estimate, is about .00033, well within the established bounds for my error. Similarly, my error with the trapezoidal estimate is about .00065, larger than my error with the midpoint approximation, and also well within the established bounds for my error.

9. Using the table, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using the trapezoidal rule and the midpoint rule. Divide the interval $[0, 6]$ into $n = 3$ equal parts. Next, if we know that $|f''(x)|$ is no bigger than 5 on the interval $[0, 6]$, then find a bound for the error of your approximations.

Time, t	0	1	2	3	4	5	6
Velocity, v	4	5	6	8	9	5	3

Trapezoidal Rule:
$\Delta x = 2$
$(2)(\frac{v(0)+v(2)}{2} + \frac{v(2)+v(4)}{2} + \frac{v(4)+v(6)}{2}) = (4+6)+(6+9)+(9+3) = 37$ meters
$|E_T| \leq \frac{K(b-a)^3}{12n^2} = \frac{5(6)^3}{12 \cdot 3^2} = \frac{5}{3}$ meters
Midpoint Rule:
$\Delta x = 2$
$(2)(v(1) + v(3) + v(5)) = (2)(5+8+5) = 36$ meters
$|E_M| \leq \frac{K(b-a)^3}{24n^2} = \frac{5(6)^3}{24 \cdot 3^2} = \frac{5}{6}$ meters

10. Consider the function $f(x) = x^2 + 3$ on the interval $[0, 1]$. Determine whether each of the following four methods of integral approximation will give an overestimate or underestimate of $\int_0^1 f(x)dx$. In each case, draw a picture to justify your answer.

(a) the left Riemann sum underestimate

(b) the right Riemann sum overestimate

(c) the trapezoidal rule overestimate

(d) the midpoint rule underestimate

11. Suppose $f(x)$ is concave up and decreasing on the interval $[0, 1]$. Suppose the approximations LEFT(100), RIGHT(100), MID(100), and TRAP(100) yield the following estimates for $\int_0^1 f(x)\,dx$: 1.10, 1.25, 1.30, and 1.50, but *not necessarily in that order*. Which estimate do you think came from which method? Between which two estimates does the exact value of the integral lie? Please explain your reasoning. 1.10: RIGHT(100), 1.25: MID(100), 1.30: TRAP(100), 1.50: LEFT(100). $\int f(x)\,dx$ must lie between MID(100) and TRAP(100).

12. Which of the following integrals can be integrated using partial fractions?

(a) $\int \frac{1}{x^4-5x^2+4}\,dx$ yes; the denominator factors as $(x^2-4)(x^2-1) = (x-2)(x+2)(x-1)(x+1)$

(b) $\int \frac{1}{x^4+1}\,dx$ If only we could factor x^4+1, then we could use partial fractions

(c) $\int \frac{1}{x^3-8}\,dx$ yes; the denominator factors as $(x-2)(x^2+2x+4)$

(d) $\int \frac{2x+1}{x^2+4}\,dx$ No. It is already in the form of one of the outputs of partial fractions. Instead, break it into two parts, $\int \frac{2x}{x^2+4}$ and $\int \frac{1}{x^2+4}$. The first can be integrated with a substitution of $u = x^2+4$, the second is an arctan function.

Make sure you can show the partial fraction decompositions.

13. For this set of problems, state which techniques are useful in evaluating the integral. You may choose from: integration by parts; partial fractions; long division; completing the square; trig substitution; or another substitution. There may be multiple answers.

   (a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ parts, or trig sub

   (b) $\displaystyle\int \frac{1}{\sqrt{6x-x^2-8}}\,dx$ completing the square; trig sub

   (c) $\displaystyle\int x\sin x\,dx$ parts

   (d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$ substitution or trig sub

   (e) $\displaystyle\int \frac{x^2}{1-x^2}\,dx$ long division; partial fractions

   (f) $\displaystyle\int (1+x^2)^{-3/2}\,dx$ trig sub

   (g) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$ substitution, followed by trig sub

   (h) $\displaystyle\int \frac{1}{1-x^2}\,dx$ partial fractions

14. A patient is given an injection of Imitrex, a migraine medicine, at a rate of $r(t) = 2te^{-2t}$ ml/sec, where t is the number of seconds since the injection started.

   (a) By using an improper integral, estimate the total quantity of Imitrex injected. $\frac{1}{2}$ ml.

   (b) What fraction of this dose has the patient received at the end of 5 seconds? About 99.95%

15. Let f be a differentiable function. Suppose that $f''(0) = 1$, $f''(1) = 2$, $f'(0) = 3$, $f'(1) = 4$, $f(0) = 5$, $f(1) = 6$. Compute $\int_0^1 f(x)f'(x)dx$. 11/2

16. For some constants A and B, the rate of production $R(t)$ of oil in a new oil well is modelled by:

$$R(t) = A + Be^{-t}\sin(2\pi t),$$

   where t is the time in years, A is the equilibrium rate, and B is the"variable" coefficient.

   (a) Find the total amount of oil produced in the first N years of operation.
   $AN - \dfrac{Be^{-N}\left(-2\pi e^N + \sin(2\pi N) + 2\pi\cos(2\pi N)\right)}{1+4\pi^2}$ (this simplifies a lot if you assume N is a whole number)

   (b) Find the average amount of oil produced per year over the first N years. it's just the above divided by N

   (c) From your answer to part (b), find the average amount of oil produced per year as $N \to \infty$. A

   (d) Looking at the function $R(t)$, explain how you might have predicted your answer to part (c) without doing any calculations. Because $Be^{-t}\sin(2\pi t) \to 0$ as $t \to \infty$

   (e) Do you think it is reasonable to expect this model to hold indefinitely? nope, oil wells eventually run dry.

17. The rate, r, at which a population of bacteria grows can be modeled by $r = te^{3t}$, where t is time in days. Find the total increase in population of bacteria after 20 days. $\frac{1+59e^{60}}{9} = 7.48649 \times 10^{26}$

18. Determine whether the following improper integrals converge or diverge.

    (a) $\displaystyle\int_1^\infty \frac{\cos^2 x}{\sqrt{x^3}}\, dx$ converges, compare to $\int_1^\infty \frac{1}{x^{3/2}}$

    (b) $\displaystyle\int_3^\infty \frac{1}{x^2 \ln x}\, dx$ converges, compare to $\int_1^\infty \frac{1}{x^2} dx$

    (c) $\displaystyle\int_1^\infty \frac{3+\sin x}{x^2}\, dx$ converges, compare to $\int_1^\infty \frac{4}{x^2} dx$

    (d) $\displaystyle\int_3^\infty \frac{5+2\sin x}{x-2}\, dx$ diverges, compare to $\int_1^\infty \frac{3}{x} dx$

    (e) $\displaystyle\int_1^\infty \frac{\ln x}{x}\, dx$ diverges, compare to $\int_1^\infty \frac{1}{x} dx$

    (f) $\displaystyle\int_1^\infty \frac{1}{x \ln x}\, dx$ diverges, evaluate the integral using substitution with $u = \ln x$. Take the limit, it is infinite.

    (g) $\displaystyle\int_{10}^\infty \frac{1}{x^2 - 9}\, dx$ converges, integrate using partial fractions, take the limit to find it is finite

19. The following integrals represent the area of some region in the xy plane. Draw a possible graph of the region, labeling the axes and giving the equation(s) of the function(s).

    (a) $\displaystyle\int_{-2}^0 (-4x)\, dx$ The triangle bounded by $y = -4x$, $x = -2$, and the x-axis.

    (b) $\displaystyle\int_{-3}^3 (-\sqrt{9-x^2})\, dx$ The semicircle bounded by $y = -\sqrt{9-x^2}$ and the x-axis.

    (c) $\displaystyle\int_1^2 3y\, dy$ The region bounded by $y = x/3$, $x = 3$, $x = 6$, and the x-axis.

    (d) $\displaystyle\int_0^1 (\sqrt{y} - y)\, dy$ The region bounded by $y = x^2$ and $y = x$

20. Using slices parallel to the base, write a definite integral representing the volume of a cone with a height of 10 cm and a base of diameter 6 cm. Depending on whether you drew the cone with vertex pointing down or pointing up you get:

$$\int_0^{10} \pi \left(\frac{3}{10}x\right)^2 dx \quad \text{or} \quad \int_0^{10} \pi \left(3 - \frac{3}{10}x\right)^2 dx$$

21. Consider the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$.

    (a) Sketch the region and find its area $\int_0^1 \sqrt{x} dx = 2/3$

    (b) Sketch the solid obtained by rotating the above region around the x-axis.

    (c) Write an integral that gives the volume of the solid $\int_0^1 \pi x\, dx = \frac{\pi}{2}$

    (d) Repeat with the same region rotated around the y-axis. $\int_0^1 \pi(1 - y^4)\, dy = 4\pi/5$

    (e) Repeat with the same region rotated around the line $x = -5$. $\pi \int_0^1 (6)^2 - (y^2 - (-5))^2 dy$

(f) Repeat with the same region rotated around the line $y = 1$. $\pi \int_0^1 1 - (1 - \sqrt{x})^2 dx$

22. Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y = x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares with one side in the xy-plane.

Since the cross sections are perpendicular to the x-axis we will have slices of thickness Δx. The volume of each slice will be given by $y\Delta x$ where $y = x - x^2$. The Riemann sum approximating the volume is then

$$\sum (x - x^2)^2 \Delta x$$

This leads us to computing the volume of the solid by the integral

$$\int_0^1 (x - x^2)^2 \, dx$$

23. Do the same thing as the previous problem except with semi-circle cross sections and then again with cross sections that are isosceles triangles of height 3. Notice that the integrals you get are just multiples of the integral from the previous problem. This means you don't have to evaluate them, instead you can just find the appropriate multiple of the answer to the previous problem!

For semi-circle cross sections we use the same method as above to get

$$\int_0^1 \frac{1}{2}\pi \left(\frac{x - x^2}{2}\right)^2 \, dx$$

as the volume of the solid. For the isosceles triangle cross-sections we get a volume of

$$\int_0^1 \frac{1}{2}(x - x^2)(3) \, dx$$

24. Suppose $f(0) = 1$, $f(1) = e$, and $f'(x) = f(x)$ for all x. Find

$$\int_0^1 e^x f'(x) dx.$$

$\frac{1}{2}(e^2 - 1)$