1. Evaluate the following integrals:

(a) [9 points] $\int x \ln x \, dx$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$

$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\,dx =$

$\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

(b) [9 points] $\int x^3 e^{x^2} \, dx$ [Hint: Make a substitution first.]

$\int x^3 e^{x^2} dx = \int x^2 e^{x^2} x\,dx$

$y = x^2 \quad dy = 2x\,dx \quad \frac{1}{2}dy = x\,dx$

$\frac{1}{2}\int y e^y dy \qquad u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

$\frac{1}{2}\int y e^y dy = \frac{1}{2}ye^y - \frac{1}{2}\int e^y dy = \frac{1}{2}ye^y - \frac{1}{2}e^y + C$

(c) [9 points] $\int_0^1 \frac{1}{(x^2+1)^{3/2}}\,dx$

$x = \tan\theta \quad x^2+1 = \sec^2\theta \quad dx = \sec^2\theta\, d\theta$

$x=0 \quad \theta = \text{Arctan}(0) \quad \theta = 0$

$x=1 \quad \theta = \text{Arctan}(1) \quad \theta = \pi/4$

$\int_0^{\pi/4} \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\,d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta = \int_0^{\pi/4} \frac{1}{\sec\theta}\,d\theta =$

$\int_0^{\pi/4} \cos\theta\, d\theta = \sin\theta \Big|_0^{\pi/4} = \sqrt{2}/2$

(d) [9 points] $\int \cos(\ln x)\,dx$ [Hint: Use integration by parts.]

$u = \cos(\ln x) \quad dv = 1\,dx$

$du = -\frac{\sin(\ln x)}{x}\,dx \quad v = x$

$\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

$\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

$u = \sin(\ln x) \quad dv = 1\,dx$

$du = \frac{\cos(\ln x)}{x}\,dx \quad v = x$

$\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$

$2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x)$

$\int \cos(\ln x)\,dx = \frac{1}{2}x\cos(\ln x) + \frac{1}{2}x\sin(\ln x) + C$

$\sin(2x) = 2\sin x \cos x$

(e) [9 points] $\int 2\sin(2x)\sin^2(x)\,dx =$

$$\int 2\cdot 2\sin x\cos x\cdot \sin^2 x\,dx = 4\int \sin^3 x\cos x\,dx$$

$$u = \sin x \quad du = \cos x\,dx$$

$$4\int u^3\,du = 4\cdot\frac{1}{4}u^4$$

Answer: $\sin^4 x + C$

(f) [9 points] $\int \frac{5x-1}{(x-1)(x-2)}\,dx$

$$\frac{5x-1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$5x-1 = A(x-2) + B(x-1)$$

$\underline{x=1}$: $4 = -A$ $\boxed{A=-4}$

$\underline{x=2}$: $\boxed{9 = B}$

$$\int \frac{5x-1}{(x-1)(x-2)}\,dx = -4\int\frac{1}{x-1}\,dx + 9\int\frac{1}{x-2}\,dx =$$

$$-4\ln|x-1| + 9\ln|x-2| + C$$

(g) [9 points] $\int \sqrt{25-x^2}\,dx$

$$x = 5\sin\theta$$
$$dx = 5\cos\theta\,d\theta$$
$$25 - x^2 = 25 - 25\sin^2\theta = 25\cos^2\theta$$

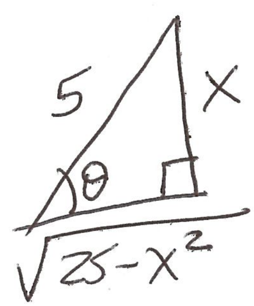
$$\int \sqrt{25\cos^2\theta}\;5\cos\theta\,d\theta =$$

$$25\int \cos^2\theta\,d\theta = \frac{25}{2}\int (1+\cos 2\theta)\,d\theta =$$

$$\frac{25}{2}\theta + \frac{25}{4}\sin 2\theta = \frac{25}{2}\theta + \frac{25}{4}\,2\sin\theta\cos\theta =$$

$$\frac{25}{2}\theta + \frac{25}{2}\sin\theta\cos\theta$$

$$\sin\theta = x/5$$



$$\theta = \arcsin\left(\frac{x}{5}\right)$$

$$\frac{25}{2}\arcsin\left(\frac{x}{5}\right) + \frac{25}{2}\,\frac{x}{5}\cdot\frac{\sqrt{25-x^2}}{5} + C$$

(h) [9 points] $\int \frac{2x^2+x-1}{(x-3)(x^2+1)}\,dx$

$$\frac{2x^2+x-1}{(x-3)(x^2+1)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+1}$$

$$2x^2+x-1 = A(x^2+1) + (Bx+C)(x-3)$$

$\underline{x=3:}$ $\quad 20 = 10A \quad A = 2$

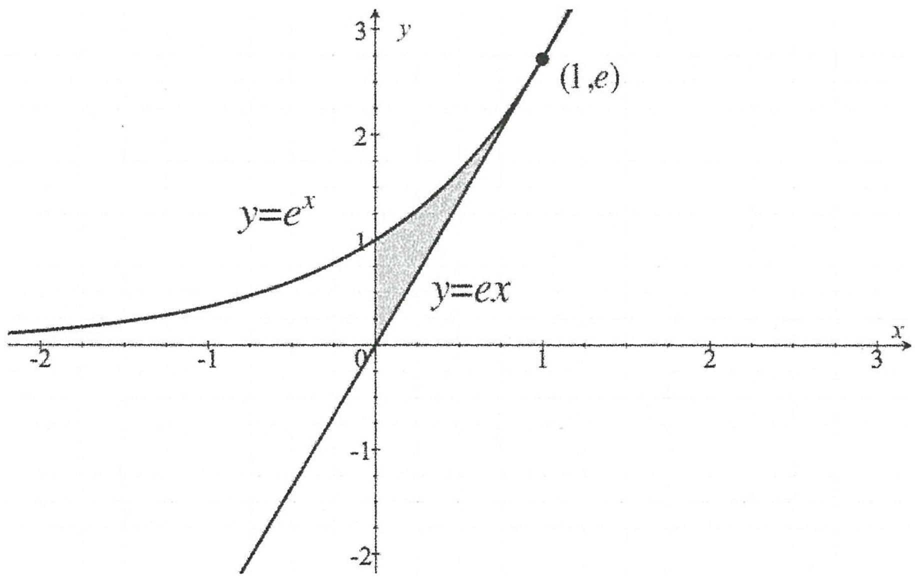
$$2x^2+x-1 = 2x^2+2+Bx^2-3Bx+Cx-3C$$

x^2: $\quad 2 = 2 + B \longrightarrow B = 0$

x: $\quad 1 = -3B + C \longrightarrow C = 1$

$$2\int \frac{1}{x-3}\,dx + \int \frac{1}{x^2+1}\,dx = 2\ln|x-3| + \arctan x + C$$

2. (12 points) Let R be the region bounded by the graphs of $y = e^x$, $y = ex$ and the y-axis.



Set up but do not compute an integral expression for the

(a) (4 points) area of R,

$$\text{Area} = \int_0^1 (e^x - ex)\,dx$$

(b) (4 points) volume of the solid of revolution obtained by rotating R about the x-axis.

$$\text{Volume} = \pi \int_0^1 [(e^x)^2 - (ex)^2]\,dx$$

3. [9 points] Determine if the integral $\int_1^\infty \frac{x}{x^2+4}\,dx$ converges or diverges by evaluating the integral.

$u = x^2+4 \qquad x=1 \quad u=5$

$du = 2x\,dx \qquad x\to\infty \quad u\to\infty$

$\frac{1}{2}du = x\,dx \qquad \frac{1}{2}\int_5^\infty \frac{1}{u}\,du =$

↑ p-integral $p=1$ (Diverges)

OR $\frac{1}{2}\lim_{b\to\infty}\int_5^b \frac{1}{u}\,du = \frac{1}{2}\lim_{b\to\infty} \ln(u)\Big|_5^b =$

$\frac{1}{2}\left[\lim_{b\to\infty} \ln(b) - \ln(5)\right] \to \infty$ Diverges

4. [9 points] Determine if the integral $\int_0^\infty \frac{1+\sin x}{e^x}\,dx$ converges or diverges.

$-1 \le \sin x \le 1 \qquad \int_0^\infty \frac{1}{e^x}\,dx =$

$\lim_{b\to\infty}\int_0^b e^{-x}\,dx = \lim_{b\to\infty} \frac{-1}{e^x}\Big|_0^b = \lim_{b\to\infty}\left[\frac{-1}{e^b}+1\right] = 1$

$\int_0^\infty \frac{1}{e^x}\,dx$ converges

$0 < \int_0^\infty \frac{1+\sin x}{e^x}\,dx < \int_0^\infty \frac{1+1}{e^x}\,dx = 2\int_0^\infty \frac{1}{e^x}\,dx$

↗ converges

So $\int_0^\infty \frac{1+\sin x}{e^x}\,dx$ converges by Comparison Theorem