SOME SERIES FULL SOLUTIONS

11a. $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ (THIS IS LIKE ALTERNATING HARMONIC IT SHOULD CONVERGE CONDITIONALLY)

$b_n = \frac{1}{n+5} > 0$

$b_{n+1} = \frac{1}{n+1+5} < \frac{1}{n+5} = b_n$, So b_n decreases

$\lim_{n\to\infty} \frac{1}{n+5} = 0.$

So by the alternating series test, $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ converges.

Now consider

$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n+5} \right| = \sum_{n=2}^{\infty} \frac{1}{n+5}$

$a_n = \frac{1}{n+5} > 0 \quad b_n = \frac{1}{n} > 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+5} = \lim_{n\to\infty} \frac{1}{1+\frac{5}{n}} = 1$

Limit is finite and nonzero, so Σa_n & Σb_n both converge or both diverge.

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series) so by The limit comparison test, $\sum_{n=2}^{\infty} \frac{1}{n+5}$ also diverges.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ does not converge absolutely, but it converges.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{n+5}$ converges conditionally

11b. $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Let $f(x) = \frac{1}{x(\ln x)^2}$. $f(x)$ is continuous and positive.

$f(x) = \left(x(\ln x)^2\right)^{-1}$, $f'(x) = -\left(x(\ln x)^2\right)^{-2}\left(x \cdot \frac{2\ln x}{x} + (\ln x)^2\right)$

$= \frac{-\left(2x\ln x + (\ln x)^2\right)}{\left(x(\ln x)^2\right)^2} < 0$, So $f(x)$ decreasing

Integral test applies.

$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{u^2}\,du = \lim_{b\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln b}$

$= \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$

The improper integral converges, so by the integral test, $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ also converges. The convergence is absolute, since the series is positive.

11c. $\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$

The series is positive so I can use the comparison test. Note that $\ln n < n$, so $\frac{1}{\ln n} > \frac{1}{n}$

$a_n = \frac{n}{(\ln n)^2} > \frac{n}{n \cdot n} = \frac{1}{n} > 0$

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (harmonic series)

so by the Term-size comparison test,

$\sum_{n=2}^{\infty} \frac{n}{(\ln n)^2}$ also diverges.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2(-3)^n}{n!}$ (Geomtric parts & factorial parts mixed with polynomial - perfect candidate for ratio test)

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2(n+1)^2(-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{2n^2(-3)^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\frac{3}{n+1} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2\cdot\frac{3}{n+1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\lim_{n\to\infty}\frac{3}{n+1} = 1\cdot 0 = 0 < 1$$

By the ratio test, $\displaystyle\sum_{n=1}^{\infty}\frac{2n^2(-3)^n}{n!}$ converges absolutely.

e) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4\cdot 2^n}{(-3)^{n+1}}+\frac{1}{2^n}\right)$

$$= \sum_{n=1}^{\infty}\frac{4\cdot 2^n}{(-3)^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{2^n}$$

The first is a geometric series with ratio $r=-\frac{2}{3}$. $|r|<1$, so it converges absolutely (Note: it converges to

$$\frac{a}{1-r} = \frac{\frac{8}{(-3)^2}}{1-\left(-\frac{2}{3}\right)} = \frac{8}{9}\cdot\frac{1}{\frac{5}{3}} = \frac{8}{15},$$ but this wasn't asked)

The second series is geometric with ratio $r=\frac{1}{2}$. $|r|<1$, so it converges absolutely by the geometric series test. (to $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$)

The sum $\displaystyle\sum_{n=1}^{\infty}\left(\frac{4\cdot 2^n}{(-3)^{n+1}}+\frac{1}{2^n}\right)$ converges absolutely.

f) $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n^3+2n}$ (This is like $\sum \frac{(-1)^n \sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{5/2}}$

so it converges absolutely)

Consider $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \sqrt{n}}{n^3+2n} \right| = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3+2n}$

$a_n = \frac{\sqrt{n}}{n^3+2n} > 0 \quad b_n = \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}} > 0.$

$a_n < b_n$ (a_n has a larger denominator than $\frac{\sqrt{n}}{n^3}$)

$\sum b_n$ converges ($p = 5/2 > 1$)

so $\sum a_n$ converges as well by term-size comparison.

$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n^3+2n}$ converges absolutely (and thus converges)

g) $\sum_{n=1}^{\infty} \frac{n+3n^5}{2n^7+3}$

$a_n = \frac{n+3n^5}{2n^7+3} > 0 \qquad b_n = \frac{n^5}{n^7} = \frac{1}{n^2} > 0$

$\sum_{n=1}^{\infty} b_n$ converges ($p = 2 > 1$)

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+3n^5}{2n^7+3} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{(n^3+3n^7)\cdot\frac{1}{n^7}}{(2n^7+3)\cdot\frac{1}{n^7}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^4}+3}{2+\frac{3}{n^7}} = \frac{3}{2}.$$ Finite & nonzero.

By the limit comparison test, $\sum_{n=1}^{\infty} a_n$ also converges.

It is already a positive-term series,

so it converges absolutely.

h. $\sum_{n=1}^{\infty} n^{\frac{1}{n}}$

Find $L = \lim_{n\to\infty} n^{\frac{1}{n}}$

$$\ln L = \lim_{n\to\infty} \ln(n^{\frac{1}{n}}) = \lim_{n\to\infty} \frac{1}{n}\cdot \ln n = \lim_{n\to\infty} \frac{\ln n}{n} \quad \left(\text{"}\frac{\infty}{\infty}\text{"}\right)$$

$$= \lim_{x\to\infty} \frac{\ln x}{x} \stackrel{L'H\hat{o}}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$\ln L = 0$ so $L = e^0 = 1$

$\lim_{n\to\infty} n^{1/n} = 1$ (not 0), so by the divergence test,

$\sum_{n=1}^{\infty} n^{1/n}$ diverges.

i. $\sum_{n=1}^{\infty} \arctan n$

$\lim_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$

So by the divergence test, $\sum_{n=1}^{\infty} \arctan n$ diverges

j. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Consider $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n^2}\right|$

$a_n = \frac{|\sin n|}{n^2} \geq 0$; $b_n = \frac{1}{n^2} > 0$

$0 \leq |\sin n| \leq 1$

$0 \leq \frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$

$\sum b_n$ converges ($p = 2 > 1$), so $\sum a_n$ converges as well

(Term-size comparison test)

So $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely (and thus converges)

12a) Show $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges by alternating Series Test.

$b_n = \frac{1}{n!} > 0 \qquad b_{n+1} = \frac{1}{(n+1)!} > \frac{1}{n!} = b_n$, so b_n is decreasing.

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(n+1)!} = 0$. So by the alternating Series Test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges.

12b) How many terms are needed to guarantee an estimate within .0001?

$|R_n| \leq b_{n+1}$ (by alt. series Rem. estimate whose hypotheses were checked above)

$b_{n+1} = \frac{1}{(n+1)!}$ By trial and error,

$\frac{1}{7!} < .000199$, not small enough

$\frac{1}{8!} < .000025$, small enough.

if $n+1=8$, $\boxed{n=7}$

$|R_n| < \frac{1}{8!} < .0001$

12c) $n=7$ suffices, $s_n = -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040}$

13. How many terms needed to guarantee an estimate of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ to within .01.

$b_n = \frac{1}{\sqrt{n}}$, decreasing, $\lim_{n\to\infty} b_n = 0$, So alt. Series Remainder Estimate applies.

$|R_n| < b_{n+1} = \frac{1}{\sqrt{n+1}} \leq .01$

$\sqrt{n+1} \geq 100$

$n+1 \geq 10000$

$n \geq 9999$