INTRODUCTION TO LINEAR ALGEBRA
MATH 3130

HOMEWORK 11
SOME SOLUTIONS

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1. Review of changing coordinates

Let V be a vector space, and let $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in V$ be two ordered bases for V. Suppose that we are given expressions for $w_1, \dots, w_n$ in terms of $v_1, \dots, v_n$. In other words, suppose that

$$\begin{array}{ccccccccc}
w_1 & = & s_{11}v_1 & + & s_{12}v_2 & + & \dots & + & s_{1n}v_n \\
w_2 & = & s_{21}v_1 & + & s_{22}v_2 & + & \dots & + & s_{2n}v_n \\
\vdots & \vdots & \vdots & & \vdots & & & \vdots & \\
w_i & = & s_{i1}v_1 & + & s_{i2}v_2 & + & \dots & + & s_{in}v_n \\
\vdots & \vdots & \vdots & & \vdots & & & \vdots & \\
w_n & = & s_{n1}v_1 & + & s_{n2}v_2 & + & \dots & + & s_{nn}v_n
\end{array}$$

for some real numbers $s_{11}, s_{12}, \dots, s_{nn}$.

If $(a_1, \dots, a_n) \in \mathbb{R}^n$ are coordinates for a vector $v \in V$ with respect to the ordered basis $v_1, \dots, v_n$ (i.e. $v = \sum_{i=1}^n a_i v_i$), then $(b_1, \dots, b_n) \in \mathbb{R}^n$ are the coordinates for the vector v with respect to the ordered basis $w_1, \dots, w_n$ (i.e. we also have $v = \sum_{i=1}^n b_i w_i$), if and only if

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \left(\begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}^T \right)^{-1} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

Conversely, if $(b_1, \dots, b_n) \in \mathbb{R}^n$ are coordinates for a vector $v \in V$ with respect to the ordered basis $w_1, \dots, w_n$ (i.e. $v = \sum_{i=1}^n b_i w_i$), then $(a_1, \dots, a_n) \in \mathbb{R}^n$ are the coordinates for the vector v with respect to

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the ordered basis $v_1, \dots, v_n$ (i.e. we also have $v = \sum_{i=1}^{n} a_i v_i$), if and only if

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}^T \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

For brevity, let us use the notation:

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

In shorthand, we then have

$$\mathbf{b} = \left(\mathbf{S}^T\right)^{-1} \mathbf{a} \quad \text{and} \quad \mathbf{a} = \mathbf{S}^T \mathbf{b}.$$

We say that:

(1) The matrix $\left(\mathbf{S}^T\right)^{-1}$ is the change of coordinates matrix from the ordered basis $v_1, \dots, v_n$ to the ordered basis $w_1, \dots, w_n$.

(2) The matrix $\mathbf{S}^T$ is the change of coordinates matrix from the ordered basis $w_1, \dots, w_n$ to the ordered basis $v_1, \dots, v_n$.

2. Problem §4.7 #1

Problem A (Problem §4.7 #1)**.** Let V be a vector space, and suppose that $v_1, v_2 \in V$ and $w_1, w_2 \in V$ are two ordered bases for V. Assume further that

$$\begin{aligned} w_1 &= 6v_1 - 2v_2 \\ w_2 &= 9v_1 - 4v_2. \end{aligned}$$

(a) Find the change of coordinates matrix from the ordered basis w_1, w_2 to the ordered basis v_1, v_2.

(b) Find the coordinates for the vector $-3w_1 + 2w_2$ in terms of the ordered basis v_1, v_2.

Solution to A. The solutions to the problem are:

(a) $\begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}$

(b) $(0, -2)$.

Here is how we can find these solutions. From what is in the previous section, the first thing to do is to find the matrix $\mathbf{S}$. Since

$$\begin{array}{rcrcr} w_1 & = & 6v_1 & - & 2v_2 \\ w_2 & = & 9v_1 & - & 4v_2, \end{array}$$

we have that

$$\mathbf{S} = \begin{pmatrix} 6 & -2 \\ 9 & -4 \end{pmatrix}$$

Thus the solution to part (a) is the matrix

$$\mathbf{S}^T = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}$$

The solution to part (b) follows directly from this. We have found that the matrix $\mathbf{S}^T$ above is the change of coordinates matrix from the ordered basis w_1, w_2, to the ordered basis v_1, v_2. The coordinates for $v = -3w_1 + 2w_2$ in terms of the ordered basis w_1, w_2 are

$$(b_1, b_2) = (-3, 2).$$

Thus the coordinates (a_1, a_2) for $v = -3w_1 + 2w_2$ in terms of the ordered basis v_1, v_2 are

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \mathbf{S}^T\mathbf{b} = \begin{pmatrix} 6 & 9 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

In other words, the answer to (b) is $(0, -2)$.

3. Problem §4.7#8

Problem B. Consider the ordered bases

$$w_1 = \begin{pmatrix} -1 \\ 8 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ -7 \end{pmatrix} \quad \text{and} \quad x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $\mathbb{R}^2$. Find the change of coordinates matrix from the ordered basis w_1, w_2 to the ordered basis x_1, x_2, and conversely, from the ordered basis x_1, x_2 to the ordered basis w_1, w_2.

Solution to B. We will solve this problem using the algorithm above. The only observation we need to make is that if we set

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to be the standard basis vectors in $\mathbb{R}^2$, then the expressions for w_1, w_2, x_1, x_2 tell us that

$$\begin{array}{rcrcr} w_1 & = & (-1)v_1 & + & 8v_2 \\ w_2 & = & 1v_1 & + & (-7)v_2 \end{array}$$

and

$$\begin{array}{rcrcr} x_1 & = & 1v_1 & + & 2v_2 \\ x_2 & = & 1v_1 & + & 1v_2. \end{array}$$

Now, for instance, to find the change of coordinates matrix from the ordered basis w_1, w_2 to the ordered basis x_1, x_2, we can first find the change of coordinates matrix from the ordered basis w_1, w_2 to the ordered basis v_1, v_2, and then find the change of coordinates matrix from the ordered basis v_1, v_2 to the ordered basis x_1, x_2.

To do this, let us focus for the moment on the ordered basis w_1, w_2. If we set

$$\mathbf{S}_W = \begin{pmatrix} -1 & 8 \\ 1 & -7 \end{pmatrix}$$

then

$$\mathbf{S}_W^T = \begin{pmatrix} -1 & 1 \\ 8 & -7 \end{pmatrix} \text{ is the change of coordinates matrix from } w_1, w_2 \text{ to } v_1, v_2,$$

and

$$\left(\mathbf{S}_W^T\right)^{-1} = \begin{pmatrix} 7 & 1 \\ 8 & 1 \end{pmatrix} \text{ is the change of coordinates matrix from } v_1, v_2 \text{ to } w_1, w_2.$$

Similarly, if we set

$$\mathbf{S}_X = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

then

$$\mathbf{S}_X^T = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \text{ is the change of coordinates matrix from } x_1, x_2 \text{ to } v_1, v_2,$$

and

$$\left(\mathbf{S}_X^T\right)^{-1} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \text{ is the change of coordinates matrix from } v_1, v_2 \text{ to } x_1, x_2.$$

Finally, we have

$$\left(\mathbf{S}_X^T\right)^{-1} \mathbf{S}_W^T$$

$$= \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 8 & -7 \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ -10 & 9 \end{pmatrix}$$

is the change of coordinates matrix from w_1, w_2 to x_1, x_2, and

$$\left(\mathbf{S}_W^T\right)^{-1} \mathbf{S}_X^T$$

$$= \begin{pmatrix} 7 & 1 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 8 \\ 10 & 9 \end{pmatrix} \text{ is the change of coordinates matrix from } x_1, x_2 \text{ to } w_1, w_2.$$

Remark 3.1. One can come up with a little faster algorithm if one remembers that in this case one is simply looking for the matrices $\left(\mathbf{S}_X^T\right)^{-1} \mathbf{S}_W^T$ and $\left(\mathbf{S}_W^T\right)^{-1} \mathbf{S}_X^T$. For instance, in the problem above, the algorithm for finding the change of coordinates matrix from the ordered basis x_1, x_2 to the ordered basis w_1, w_2 would be the following. Recall we are given ordered bases

$$w_1 = \begin{pmatrix} -1 \\ 8 \end{pmatrix}, w_2 = \begin{pmatrix} 1 \\ -7 \end{pmatrix} \quad \text{and} \quad x_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $\mathbb{R}^2$. To find the change of coordinates matrix from the ordered basis x_1, x_2 to the ordered basis w_1, w_2, the algorithm says to enter the basis vectors in column form into a matrix:

$$\left(\begin{array}{cc|cc} -1 & 1 & 1 & 1 \\ 8 & -7 & 2 & 1 \end{array}\right)$$

In our notation above, this is the matrix

$$\left(\begin{array}{c|c} \mathbf{S}_W^T & \mathbf{S}_X^T \end{array}\right)$$

The algorithm then asks you to row reduce this matrix. This of course gives the matrix

$$\left(\begin{array}{c|c} Id & \left(\mathbf{S}_W^T\right)^{-1} \mathbf{S}_X^T \end{array}\right) = \left(\begin{array}{cc|cc} 1 & 0 & 9 & 8 \\ 0 & 1 & 10 & 9 \end{array}\right)$$

and the 2×2 matrix on the right is the matrix we are interested in.