INTRODUCTION TO LINEAR ALGEBRA
MATH 3130

HOMEWORK 1

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1. Vector spaces

Recall the definition of a real vector space.

Definition 1.1. *A* ***real vector space*** *consists of a triple* $(V, +, \cdot)$*, where* V *is a set, and* $+ : V \times V \to V$ *and* $\cdot : \mathbb{R} \times V \to V$ *are maps, satisfying the following properties:*

(1) *(Group laws)*

(a) *(Additive identity) There exists an element* $O \in V$ *such that for all* $v \in V$, $v + O = v$*;*

(b) *(Additive inverse) For each* $v \in V$ *there exists an element* $-v \in V$ *such that* $v + (-v) = O$*;*

(c) *(Associativity of addition) For all* $v_1, v_2, v_3 \in V$,

$$(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3);$$

(2) *(Abelian property)*

(a) *(Commutativity of addition) For all* $v_1, v_2 \in V$,

$$v_1 + v_2 = v_2 + v_1;$$

(3) *(Module conditions)*

(a) *For all* $\lambda \in \mathbb{R}$ *and all* $v_1, v_2 \in V$,

$$\lambda \cdot (v_1 + v_2) = (\lambda \cdot v_1) + (\lambda \cdot v_2)$$

(b) *For all* $\lambda_1, \lambda_2 \in \mathbb{R}$*, and all* $v \in V$,

$$(\lambda_1 + \lambda_2) \cdot v = (\lambda_1 \cdot v) + (\lambda_2 \cdot v).$$

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(c) *For all* $\lambda_1, \lambda_2 \in \mathbb{R}$*, and all* $v \in V$*,*

$$(\lambda_1 \lambda_2) \cdot v = \lambda_1 \cdot (\lambda_2 \cdot v).$$

(d) *For all* $v \in V$*,*

$$1 \cdot v = v.$$

In the above, for all $\lambda \in \mathbb{R}$ *and all* $v, v_1, v_2 \in V$ *we have denoted* $+(v_1, v_2)$ *by* $v_1 + v_2$ *and* $\cdot(\lambda, v)$ *by* $\lambda \cdot v$*.*

In addition, for brevity, we will often write λv for $\lambda \cdot v$.

Remark 1.2. Recall that in class we discussed the example of the vector space $(\mathbb{R}^n, +, \cdot)$. By definition,

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \ : \ x_i \in \mathbb{R}, \ 1 \leq i \leq n\}.$$

The map $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by the rule $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$ for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}^n$. The map $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by the rule $\lambda \cdot (x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$ for all $\lambda \in \mathbb{R}$ and $(x_1, \dots, x_n) \in \mathbb{R}^n$.

Exercise A. Let $(V, +, \cdot)$ be a vector space. Show that if $v \in V$ satisfies $v' + v = v'$ for all $v' \in V$, then $v = O$, the additive identity.

Exercise B. Show the following properties hold for all $v, v_1, v_2 \in V$ and all $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$.

(1) $0v = O$.

(2) $\lambda O = O$.

(3) $(-\lambda)v = -(\lambda v) = \lambda(-v)$.

(4) If $\lambda v = O$, then either $\lambda = 0$ or $v = O$.

(5) If $\lambda v_1 = \lambda v_2$, then either $\lambda = 0$ or $v_1 = v_2$.

(6) If $\lambda_1 v = \lambda_2 v$, then either $\lambda_1 = \lambda_2$ or $v = O$.

(7) $-(v_1 + v_2) = (-v_1) + (-v_2)$.

(8) $v + v = 2v$, $v + v + v = 3v$, and in general $\sum_{i=1}^{n} v = nv$.

Exercise C. Consider the set of functions from $\mathbb{R}$ to itself. Let us denote this set by $\mathrm{Map}(\mathbb{R}, \mathbb{R})$. Define addition and multiplication maps

$$+ : \mathrm{Map}(\mathbb{R}, \mathbb{R}) \times \mathrm{Map}(\mathbb{R}, \mathbb{R}) \to \mathrm{Map}(\mathbb{R}, \mathbb{R})$$

and

$$\cdot : \mathbb{R} \times \mathrm{Map}(\mathbb{R}, \mathbb{R}) \to \mathrm{Map}(\mathbb{R}, \mathbb{R})$$

in the following way. For all $f, g \in \mathrm{Map}(\mathbb{R},\mathbb{R})$, set $f+g$ to be the function defined by $(f+g)(x) = f(x)+g(x)$ for all $x \in \mathbb{R}$. For all $\lambda \in \mathbb{R}$ and all $f \in \mathrm{Map}(\mathbb{R},\mathbb{R})$, set $\lambda \cdot f$ to be the function defined by $(\lambda \cdot f)(x) = \lambda f(x)$ for all $x \in \mathbb{R}$. *Show that* $(\mathrm{Map}(\mathbb{R},\mathbb{R}), +, \cdot)$ *is a vector space.*

2. Sub-vector spaces

Recall the definition of a sub-vector space.

Definition 2.1. *Let* $(V, +, \cdot)$ *be a (real) vector space. A (real)* ***sub-vector space*** *of* $(V, +, \cdot)$ *is a (real) vector space* $(V', +', \cdot')$ *such that* $V' \subseteq V$ *and such that for all* $v', v_1', v_2' \in V'$ *and all* $\lambda \in \mathbb{R}$,

$$v_1' +' v_2' = v_1' + v_2' \qquad \text{and} \qquad \lambda \cdot' v' = \lambda \cdot v'.$$

Definition 2.2. *If* $(V, +, \cdot)$ *is a vector space, and* $V' \subseteq V$ *is a subset, we say that* V' *is* ***closed under*** $+$ ***(resp. closed under*** $\cdot$***)*** *if for all* $v_1', v_2' \in V'$ *(resp. for all* $\lambda \in \mathbb{R}$ *and all* $v' \in V'$*) we have* $v_1' + v_2' \in V'$ *(resp.* $\lambda \cdot v' \in V'$*). In this case, we define* $+|_{V'} : V' \times V' \to V'$ *(resp.* $\cdot|_{V'} : \mathbb{R} \times V' \to V'$*) to be the map given by* $v_1' +|_{V'} v_2' = v_1' + v_2'$ *(resp.* $\lambda \cdot|_{V'} v' = \lambda \cdot v'$*), for all* $v_1', v_2' \in V'$ *(resp. for all* $\lambda \in \mathbb{R}$ *and all* $v' \in V'$*).*

Remark 2.3. Note that if $(V', +', \cdot')$ is a sub-vector space of $(V, +, \cdot)$, then V' is closed under $+$ and $\cdot$.

Exercise D. Show that if a non-empty subset $V' \subseteq V$ is closed under $+$ and $\cdot$, then $(V', +|_{V'}, \cdot|_{V'})$ is a sub-vector space of $(V, +, \cdot)$.

Exercise E. Show that if $(V', +', \cdot')$ is a sub-vector space of a vector space $(V, +, \cdot)$, then the additive identity element $O' \in V'$ is equal to the additive identity element $O \in V$.

Exercise F. Recall the vector space $(\mathrm{Map}(\mathbb{R},\mathbb{R}), +, \cdot)$ from Exercise C. In this exercise, show that the subsets of $\mathrm{Map}(\mathbb{R},\mathbb{R})$ listed below are closed under $+$ and $\cdot$, and so define sub-vector spaces of $(\mathrm{Map}(\mathbb{R},\mathbb{R}), +, \cdot)$.

(1) **Problem removed.**
(2) The set of all polynomial functions.
(3) The set of all polynomial functions of degree less than n.
(4) The set of all functions that are continuos on an interval $(a, b) \subseteq \mathbb{R}$.
(5) The set of all functions differentiable at a point $a \in \mathbb{R}$.
(6) The set of all functions differentiable on an interval $(a, b) \subseteq \mathbb{R}$.
(7) The set of all functions with $f(1) = 0$.
(8) The set of all solutions to the differential equation $f'' + af' + bf = 0$ for some $a, b \in \mathbb{R}$.

Exercise G. In this exercise, show that the subsets of $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ listed below are NOT closed under $+$ and $\cdot$, and so do not define sub-vector spaces of $(\mathrm{Map}(\mathbb{R}, \mathbb{R}), +, \cdot)$.

(1) Fix a real number $a \neq 0$. The set of all functions with $f(1) = a$.

(2) The set of all solutions to the differential equation $f'' + af' + bf = c$ for some $a, b, c \in \mathbb{R}$ with $c \neq 0$.

3. Linear maps

Recall the definition of a linear map.

Definition 3.1. *Let $(V, +, \cdot)$ and $(V', +', \cdot')$ be vector spaces. A* ***linear map*** *$F : (V, +, \cdot) \to (V', +', \cdot')$ is a map of sets*

$$f : V \to V'$$

such that for all $\lambda \in \mathbb{R}$ and $v, v_1, v_2 \in V$,

$$f(v_1 + v_2) = f(v_1) +' f(v_2) \quad \text{and} \quad f(\lambda \cdot v) = \lambda \cdot' f(v).$$

Note that we will frequently use the same letter for the linear map and the map of sets. Recall that $(V, +, \cdot)$ is called the **source** (or domain) of the linear map and $(V', +', \cdot')$ is called the **target** (or codomain) of the linear map. The set $f(V) \subseteq V'$ is called the **image** (or range) of f.

Exercise H. Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of vector spaces. Show that the image of f is closed under $+', \cdot'$, and so defines a sub-vector space of the target $(V', +', \cdot')$.

Exercise I. Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of vector spaces. Show that $f(O) = O'$.

Exercise J. Show that the following maps of sets define linear maps of the vector spaces.

(1) Let $(V, +, \cdot)$ be a vector space. The identity map $f : V \to V$ given by $f(v) = v$ for all $v \in V$. This linear map will frequently be denoted by Id_V.

(2) Let $(V, +, \cdot)$ be a vector space. The zero map $f : V \to V$ given by $f(v) = O$ for all $v \in V$. This linear map will frequently be denoted by O_V.

(3) Let $(V, +, \cdot)$ be a vector space and let $\alpha \in \mathbb{R}$. The multiplication map $f : V \to V$ given by $f(v) = \alpha \cdot v$ for all $v \in V$. This linear map will frequently be denoted by $\alpha \, \mathrm{Id}_V$.

(4) Let $(V, +, \cdot) = \mathbb{R}^n$, let $(V', +', \cdot') = \mathbb{R}^m$, and let $a_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. Define a map $f : \mathbb{R}^n \to \mathbb{R}^m$ by

$$f(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{ij} x_j, \ldots, \sum_{j=1}^{n} a_{mj} x_j \right).$$

(5) Let $(V, +, \cdot)$ be the vector space of all differentiable real functions. Let $(V', +', \cdot')$ be the vector space of all real functions. The map $f : (V, +, \cdot) \to (V', +', \cdot')$ that sends a differentiable function g to its derivative g'.

(6) Let $(V, +, \cdot)$ be the vector space of all continuous real functions. The map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the function in V determined by

$$f(g)(x) := \int_a^x g(t)dt.$$

Show also that f in fact sends V to V.

Recall the definition of the kernel of a linear map.

Definition 3.2. *Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of vector spaces. The* ***kernel of*** *f (or Null space of f), denoted* $\ker(f)$*, is the set*

$$\ker(f) := f^{-1}(O') = \{v \in V : f(v) = O'\}.$$

Exercise K. Let $f : (V, +, \cdot) \to (V', +', \cdot')$ be a linear map of vector spaces. Show that $\ker(f)$ is a sub-vector space of $(V, +, \cdot)$.

Exercise L. Find the kernel of each of the linear maps listed below (see Problem J).

(1) The linear map Id_V.

(2) The linear map O_V.

(3) The linear map $\alpha\,\mathrm{Id}_V$.

(4) Let $(V, +, \cdot) = \mathbb{R}^n$, let $(V', +', \cdot') = \mathbb{R}^m$, and let $a_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. The linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$f(x_1, \ldots, x_n) = \left(\sum_{j=1}^n a_{1j}x_j, \ldots, \sum_{j=1}^n a_{ij}x_j, \ldots, \sum_{j=1}^n a_{mj}x_j \right).$$

(5) Let $(V, +, \cdot)$ be the vector space of all differentiable real functions. Let $(V', +', \cdot')$ be the vector space of all real functions. The linear map $f : (V, +, \cdot) \to (V', +', \cdot')$ that sends a differentiable function g to its derivative g'.

(6) Let $(V, +, \cdot)$ be the vector space of all continous real functions. The linear map $f : (V, +, \cdot) \to (V, +, \cdot)$ that sends a function $g \in V$ to the function in V determined by

$$f(g)(x) := \int_a^x g(t)dt.$$