SAMPLE MIDTERM I ANALYSIS 1

MATH 3100

Friday October 4, 2013

Name

Please answer all of the questions, and show your work. All solutions must be explained clearly to receive credit.

1	2	3	4	5	6	7	
10	10	10	10	10	10	10	Total

Date: September 29, 2013.

1 10 points

Let S and T be non-empty sets. Let Map(S,T) be the set of maps from S to T.
(a). What is the definition of a map from S to T.

1.(b). If T is uncountable, show that Map(S,T) is uncountable.

1.(c). If S is infinite and T has at least 2 elements, show that Map(S,T) uncountable.

- **2.** Let K be a field, and let \leq be a total ordering on K.
- **2.(a).** What conditions on \leq must be satisfied for (K, \leq) to be an ordered field.

2.(b). Assume (K, \leq) is an odered field. Use the definition of an ordered field to show that if $x_1, x_2, x_3 \in K$, $x_1 > x_2$ and $x_3 < 0$, then $x_1x_3 < x_1x_2$. (You may use facts about fields (that are not ordered) without proof.)

2.(c). Assume (K, \leq) is an ordered field. Use Part (b) to show that $x^2 \geq 0$ for all $x \in K$. (You may use facts about fields (that are not ordered) without proof.)

3	
10	points

3. Let (K, \leq) be an ordered field.

3.(a). One can show (although you do NOT need to do so here) that for $a \in K$, $|x| \leq a$ if and only if $-a \leq x \leq a$. Use this to show that for $x_1, x_2 \in K$,

$$|x_1 + x_2| \le |x_1| + |x_2|.$$

3.(b). Use part (a) to show by induction that for any $n \in \mathbb{N}$, if $x_1, \ldots, x_n \in K$, then $|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|$.

4	
10	points

- **4.** Let $n \in \mathbb{N}$. Suppose that $x, y \in \mathbb{R}, x, y \ge 0$. Show *either* of the following (your choice):

 - (1) If $x^n < y$, then there exists $t \in \mathbb{R}$ with x < t such that $x^n < t^n < y$. (2) If $x^n > y$, then there exists $t \in \mathbb{R}$ with x > t > 0 such that $x^n > t^n > y$.

5 10 points

5. Consider the set

$$S = \left(\bigcup_{n \in \mathbb{N}} \{1/n\}\right) \cup (2,3]$$

Answer the following questions. You do NOT need to prove your answers. 5.(a). What is sup S, the supremum of S (if it exists)?

- **5.(b).** What is $\inf S$, the infemum of S (if it exists)?
- **5.(c).** What is S° , the interior of S?
- **5.(d).** What is \overline{S} , the closure of S?
- **5.(e).** What is ∂S , the boundary of S?
- **5.(f).** What is L(S), the set of limit points of S?
- **5.(g).** What is the set of isolated points of S?

6	
10	points

6.(a). Show that if a subset $S \subseteq \mathbb{R}$ of the reals is not an interval, then there are open subsets $U \subseteq \mathbb{R}$ and $U' \subseteq \mathbb{R}$ with $U \cap U' = \emptyset$, $U \cap S \neq \emptyset$, and $U' \cap S \neq \emptyset$, such that $S \subseteq U \cup U'$.

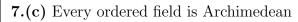
6.(b). Show that if a subset $S \subseteq \mathbb{R}$ of the reals is an interval, then there do not exist open subsets $U \subseteq \mathbb{R}$ and $U' \subseteq \mathbb{R}$ with $U \cap U' = \emptyset$, $U \cap S \neq \emptyset$, and $U' \cap S \neq \emptyset$, such that $S \subseteq U \cup U'$. [Hint: Let $a \in U \cap S$ and $b \in U' \cap S$. Assume WLOG that a < b. Then set $T = \{t \in [a, b] : t \in U\}$.]

7. True or False. You do NOT need to justify your answer.



7.(a) The set of subsets of a countable set is countable.

7.(b) If K is a field, with a total order <, and $k^2 < 0$ for some $k \in K$, then K is not an ordered field.



7.(d) Every non-empty subset of \mathbb{N} contains a minimum.

7.(e) If x and y are irrational then xy is irrational.

7.(f) A countable union of countable sets is countable.

7.(g) Every compact subset of \mathbb{R} has a limit point.



7.(h) The closure of a set is closed.

7.(i) Every Archimedean ordered field is complete.

7.(j) If $S \subseteq \mathbb{R}$, then the set of limit points L(S) is closed.