SAMPLE FINAL
ANALYSIS 1

MATH 3100

Saturday, December 14, 2013

Name	

Please answer all of the questions, and show your work.
All solutions must be explained clearly to receive credit.

1	2	3	4	5	6	7	
30	30	30	30	30	30	20	Total

Date: December 10, 2013.

1
30 points

1. For this problem let $D \subseteq \mathbb{R}$.

1.(a). State the definition of continuity for a function $f : D \to \mathbb{R}$.

1.(b). State the definition of uniform continuity for a function $f : D \to \mathbb{R}$.

1.(c). Prove that the product of any two uniformly continuous functions on a bounded interval is uniformly continuous. (You may use standard theorems to prove this.)

2
30 points

2.(a). State the Intermediate Value Theorem.

2.(b). Prove that the equation $x^2 = \cos x$ has at least one solution on the interval $[0, \pi/2]$.

3
30 points

3.(a). State the definition of differentiability.

3.(b). Show that

$$f(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable at $x = 0$.

4
30 points

4.(a). State the Mean Value Theorem.

4.(b). Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, and differentiable on (a, b). Assume that f' is increasing on (a, b). Show that for all $x \in [a, b]$

$$f(x) \leq \left(\frac{f(b) - f(a)}{b - a} \right) (x - a) + f(a).$$

In other words, show that the graph of f lies (at or) below the secant line through $(a, f(a))$ and $(b, f(b))$.

5
30 points

5.(a). State the definition of a Riemann integrable function.

5.(b). Show that the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

is not Riemann integrable on $[0,\ 1]$.

6
30 points

6. For this problem let f,g be functions that are Riemann integrable over $[a, b]$.

6.(a). Show that for any $t \in \mathbb{R}$, $t \neq 0$, we have

$$\left| \int_a^b fg \right| \leq \frac{1}{2} \left(t^2 \int_a^b f^2 + \frac{1}{t^2} \int_a^b g^2 \right)$$

[Hint: consider the integral $\int_a^b (tf \pm \frac{1}{t} g)^2$].

6.(b). Use the previous problem to show that if $\int_a^b f^2 = 0$, then

$$\int_a^b fg = 0.$$

7
20 points

7. True or False. *You do NOT need to justify your answer.*

☐ **7.(a)** The set of finite subsets of a countably infinite set is countable.

☐ **7.(b)** Every ordered field is contained in a complete ordered field.

☐ **7.(c)** Every compact subset of $\mathbb{R}$ has a limit (accumulation) point.

☐ **7.(d)** The interior S° of a set $S \subseteq \mathbb{R}$ is a subset of S, i.e. $S^\circ \subseteq S$.

☐ **7.(e)** The closure of a subset contains its boundary.

☐ **7.(f)** A bounded sequence of real numbers contains a convergent subsequence.

☐ **7.(g)** Let (s_n) be a bounded sequence. Then for every $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $s_m < (\limsup(s_n)) + \epsilon$ for all $m \geq N_\epsilon$.

☐ **7.(h)** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $U \subseteq \mathbb{R}$ is open, then $f(U)$ is open.

☐ **7.(i)** Let U be an open subset of $\mathbb{R}$. If $f : U \to \mathbb{R}$ is differentiable at $c \in U$, then f is continuous at c.

☐ **7.(j)** If $f : (a, b) \to \mathbb{R}$ is bounded and continuous, then it is uniformly continuous.

☐ **7.(k)** The product of any two uniformly continuous functions is uniformly continuous.

7.(l) If $f : (a, b) \to \mathbb{R}$ is differentiable and strictly increasing on the interval (a, b) (i.e. $x < y$ implies that $f(x) < f(y)$), then $f'(x) > 0$ for all $x \in (a, b)$.

7.(m) If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), and also satisfies $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

7.(n) If $f : [a, b] \to \mathbb{R}$ is differentiable, and $f'(a) \leq k \leq f'(b)$ for some $k \in \mathbb{R}$, then there exists $c \in (a, b)$ such that $f'(c) = k$.

7.(o) Let $f : [a, b] \to \mathbb{R}$. If $|f|$ is Riemann integrable then, f is Riemann integrable.

7.(p) Let $f : [a, b] \to \mathbb{R}$. If f is Riemann integrable then, f^2 is Riemann integrable.

7.(q) If f^2 and g^2 are Riemann integrable on $[a, b]$, then fg is Riemann integrable on $[a, b]$.

7.(r) If f is Riemann integrable on $[a, b]$, and $\int_a^b f^2 = 0$, then $f = 0$.

7.(s) If f is Riemann integrable on $[a, b]$, then the function $F(x) := \int_a^x f$ defined on $[a, b]$ is differentiable and $F'(x) = f(x)$ for all $x \in [a, b]$.

7.(t) If f is defined on $[a, b]$ and has an anti-derivative F on $[a, b]$, then f is Riemann integrable and $F(x) = \int_a^x f$ for all $x \in [a, b]$.