MATH 2300 – CALCULUS II – UNIVERSITY OF COLORADO

Spring 2011 – Final exam review problems: ANSWER KEY

1. Find $f_x(1,0)$ for $f(x,y) = \dfrac{xe^{\sin(x^2y)}}{(x^2+y^2)^{3/2}}$. -2

2. Consider the solid region $\mathcal{W}$ situated above the region $0 \le x \le 2$, $0 \le y \le x$, and bounded above by the surface $z = e^{x^2}$.

   An equivalent description for the region $0 \le x \le 2$, $0 \le y \le x$ is: $0 \le y \le 2$, $y \le x \le 2$.

   (a) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $y = a$, where a is a constant in $[0,2]$. Can you evaluate this integral (as an expression depending on a)?

   In each cross-sectional plane $y = a$, the interval for x is given by $a \le x \le 2$, so the area of the cross-sections is $\int_a^2 f(x,a)\,dx = \int_a^2 e^{x^2}\,dx$, which can't be evaluated directly.

   (b) Write an integral that evaluates the area of each cross-section of $\mathcal{W}$ with vertical planes $x = b$, where b is a constant in $[0,2]$. Can you evaluate this integral (as an expression depending on b)?
   In each cross-sectional plane $x = b$, the interval for y is given by $0 \le y \le b$, so the area of the cross-sections is $\int_0^b f(b,y)\,dy = \int_0^b e^{b^2}\,dy = be^{b^2}$.

   (c) Write a convenient double integral that evaluates the volume of the region $\mathcal{W}$, and evaluate it.

$$\int\int_{\mathcal{R}} z\,dA = \int_0^2\int_0^x e^{x^2}\,dy\,dx = \int_0^2\int_y^2 e^{x^2}\,dx\,dy$$

   The previous remarks indicate that the first option is the only convenient one, so the volume of the solid region can be calculated as:

$$\int_0^2\int_0^x e^{x^2}\,dy\,dx = \int_0^2 ye^{x^2}\Big|_{y=0}^{y=x}\,dx = \int_0^2 xe^{x^2}\,dx = \frac{e^{x^2}}{2}\Bigg|_0^2 = \frac{e^4-1}{2}$$

   (d) Let

$$f(x,y) = \sqrt{x^2+y^2}.$$

   (a) Find $f_x(x,y)$. $x/\sqrt{x^2+y^2}$.

   (b) Let $g(x) = f_x(x,0)$. Does $\lim_{x\to 0} g(x)$ exist? If so, evaluate it. If not, explain why not. Hint: Note that $f(x,0) = \sqrt{x^2} = |x|$. No.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{x}{\sqrt{x^2}} = \lim_{x\to 0}\frac{x}{|x|}.$$

   The limit does not exist because

$$\frac{x}{|x|} = \begin{cases} \dfrac{x}{(-x)} = -1 & \text{if } x < 0,\\ \dfrac{x}{x} = 1 & \text{if } x > 0.\end{cases}$$

3. Evaluate:

(a)

$$\int_1^2 \int_y^{y^2} (x^2+y)\,dx\,dy.$$

$\frac{87}{14}$

(b)

$$\int_0^1 \int_{e^y}^{e} \frac{1+e^x}{\ln x}\,dx\,dy.$$

$\frac{1}{18}\left(4e^3+9e-16\right)$

4. (a) Set up a double integral that corresponds to the mass of the semidisk

$$D = \{(x,y)\colon 0 \le y \le \sqrt{4-x^2}\},$$

if the density at each of its points is given by the function $\delta(x,y) = x^2y$. $\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} x^2y\,dy\,dx$

(b) Find the mass of the above semidisk. $\frac{64}{15}$

5. The power series $\sum C_n x^n$ diverges at $x=7$ and converges at $x=-3$. At $x=-9$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

6. The power series $\sum C_n (x-5)^n$ converges at $x=-5$ and diverges at $x=-10$. At $x=11$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

7. The power series $\sum C_n x^n$ diverges at $x=7$ and converges at $x=-3$. At $x=-4$, the series is

(a) Conditionally convergent
(b) Absolutely convergent
(c) Divergent
(d) Cannot be determined.

8. In order to determine if the series

$$\sum_{k=1}^{\infty} \frac{\sqrt{k+1}}{k^2+1}$$

converges or diverges, the limit comparison test can be used. Decide which series provides the best comparison.

(a) $\sum_{k=1}^{\infty} \frac{1}{k}$

(b) $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

(c) $\sum_{k=1}^{\infty} \frac{1}{k^2}$

9. Decide whether the following statements are true or false. Give a brief justification for your answer.

   (a) If f is continuous for all x and $\int_0^{\infty} f(x)dx$ converges, then so does $\int_a^{\infty} f(x)dx$ for all positive a. TRUE

   (b) if $f(x)$ is continuous and positive for $x > 0$ and if $\lim_{x\to\infty} f(x) = 0$, then $\int_0^{\infty} f(x)dx$ converges. FALSE

   (c) If $\int_0^{\infty} f(x)dx$ and $\int_0^{\infty} g(x)dx$ both converge then $\int_0^{\infty}(f(x)+g(x))dx$ converges. TRUE

   (d) If $\int_0^{\infty} f(x)dx$ and $\int_0^{\infty} g(x)dx$ both diverge then $\int_0^{\infty}(f(x)+g(x))dx$ diverges. FALSE

10. For this problem, state which of the integration techniques you would use to evaluate the integral, but **DO NOT** evaluate the integrals. If your answer is **substitution**, also list u and du; if your answer is **integration by parts**, also list u, dv, du and v; if your answer is **partial fractions**, set up the partial fraction decomposition, but do not solve for the numerators; if your answer is **trigonometric substitution**, write which substitution you would use.

    (a) $\int \tan x dx$ Use substitution $w = \cos x$, so that $dw = -\sin x$

    (b) $\int \frac{dx}{x^2-9}$ Factor out the denominator as $x^2 - 9 = (x-3)(x+3)$, and use partial fractions to rewrite the integrand as $\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$.

    (c) $\int e^x \cos x dx$ Start by using an integration by parts, with $u = e^x$ and $dv = \cos x dx$, so that $du = e^x\, dx$ and $v = \sin x$. (Then use another integration by parts with $U = e^x$ and $dV = \sin x dx$, and solve the resulting equation for the integral.)

    (d) $\int \frac{\sqrt{9-x^2}}{x^2} dx$ Start by using a trig substitution $x = 3\sin t$, so that $dx = 3\cos t dt$.

    (e) $\int \frac{\sin(\ln x)}{x} dx$ Substitute $w = \ln x$, so that $dw = \frac{1}{x} dx$.

    (f) $\int x^{3/2} \ln x dx$ Use integration by parts, with $u = \ln x$ and $dv = x^{3/2} dx$, so that $du = \frac{1}{x}$ and $v = \frac{2}{5} x^{5/2}$.

    (g) $\int \frac{1}{\sqrt{x^2+4}} dx$ Trig substitution $x = 2\tan t$, so that $dx = 2\sec^2 t$.

    (h) $\int \frac{e^x+1}{e^x+x} dx$ The numerator is the derivative of the denominator. Use substitution $w = e^x + x$, so that $dw = (e^x+1)dx$.

11. If

$$\sum_{n=1}^{\infty} a_n(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{4} - \frac{(x-1)^4}{8} + \frac{(x-1)^5}{16} - \cdots,$$

then the correct formula for a_n is: Oops, in the original version of this revew sheet, all of the possible answers below had a factor of $(x-1)^n$. Please ignore those.

(a) $a_n = \frac{(-1)^n}{2n}$

(b) $a_n = \frac{(-1)^{n+1}}{2^n}$

(c) $a_n = \frac{(-1)^n}{2^n}$

(d) $a_n = \frac{(-1)^{n+1}2}{2^n}$

12. **True or False?** If $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=0}^{\infty} a_n$ converges.

(a) True

(b) False

13. **True or False?** If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges.

(a) True

(b) False

14. True/False: If $\sum a_n$ is convergent, then the power series $\sum a_n x^n$ has convergence radius at least $R = 1$. True

15. If one uses the Taylor polynomial $P_3(x)$ of degree $n = 3$ to approximate $\sin x = \sum_0^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ at $x = 0.1$, would one get an overestimate or an underestimate? underestimate

16. Suppose that x is positive but very small. Arrange the following in **increasing** order:

$$x,\ \ \sin x,\ \ \ln(1+x),\ \ 1-\cos(x),\ \ e^x - 1,\ \ x\sqrt{1-x}.$$

$$1 - \cos x < x\sqrt{1-x} < \ln(1+x) < \sin x < e^x - 1$$

17. (a) Find the Taylor series about 0 for $f(x) = x^2 e^{x^2}$. $x^2 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+2}}{n!}$

(b) Is this function even or odd? Justify your answer. even

(c) Find $f^{(3)}(0)$ and $f^{(6)}(0)$. $f^{(3)}(0) = 0$, $f^{(6)}(0) = \frac{6!}{2!}$

18. Suppose that ibuprofen is taken in 200mg doses every six hours, and that all 200mg are delivered to the patient's body immediately when the pill is taken. After six hours, 12.5% of the ibuprofen remains. Find expressions for the amount of ibuprofen in the patient immediately before and after the n^{th} pill taken. Include work; without work, you may receive no credit.

Just before the nth dose:

$$200(0.125 + 0.125^2 + 0.125^3 + \cdots + 0.125^{n-1}) = 200 \sum_{k=0}^{n-1} 0.125^k = 200 \cdot \frac{1 - 0.125^n}{1 - 0.125}$$

Just after the nth dose:

$$200 + 200 \cdot \frac{1 - 0.125^n}{1 - 0.125}$$

19. Find an equation for a sphere if one of its diameters has endpoints $(2, 1, 4)$ and $(4, 3, 10)$. $(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$

20. A cube is located such that its top four corners have the coordinates $(-1, -2, 2)$, $(-1, 3, 2)$, $(4, -2, 2)$, and $(4, 3, 2)$. Give the coordinate of the center of the cube. $(1.5, 0.5, -0.5)$

21. Find the equation of the largest sphere with center $(5, 4, 9)$ contained in the first octant. $(x-5)^2 + (y-4)^2 + (z-9)^2 = 16$

22. Evaluate the double integral (using the most convenient method):

    (a) $\int_0^1 \int_{e^y}^{e} \frac{x}{\ln x}\, dx\, dy$ $\frac{e^2 - 1}{2}$

    (b) $\int_0^1 \int_x^1 \frac{y^2}{1 + y^4}\, dy\, dx$ $\frac{\ln 2}{4}$

23. The following sum of double integrals describes the mass of a thin plate $\mathcal{R}$ in the xy-plane, of density $\delta(x, y) = x + y$:

$$\text{mass} = \int_{-4}^{0} \int_{0}^{2x+8} \delta(x, y)\ dy\ dx + \int_{0}^{4} \int_{0}^{-2x+8} \delta(x, y)\ dy\ dx$$

    (a) Describe the thin plate (shape, intersections with the coordinate axes). It's an isosceles triangle with base along the x axis from -4 to 4, and apex on the y axis at $(0, 4)$.

    (b) Write an expression for the mass of the plate as only one double integral.

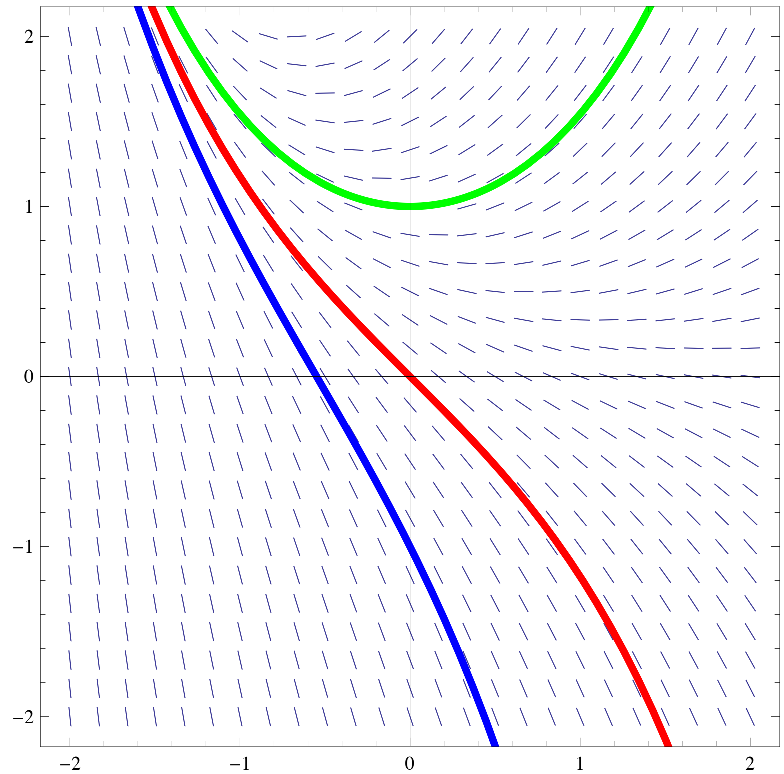
$$\int_0^4 \int_{(y-8)/2}^{(8-y)/2} (x + y)\, dx\, dy$$

    (c) Calculate the area and the mass of the plate. 256/3

24. If one uses the Taylor polynomial $P_5(x)$ of degree $n = 5$ to approximate $e^x = \sum_0^\infty \frac{x^n}{n!}$ at $x = -0.2$, would one get an overestimate or an underestimate? underestimate

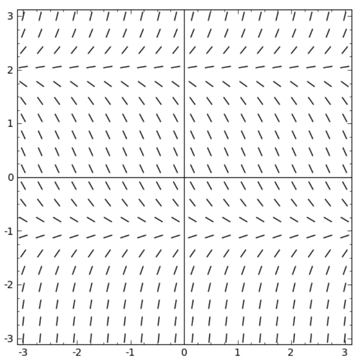
25. A slope field for the differential equation $y' = y - e^{-x}$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions. Make sure to label each sketched graph.

    (a) $y(0) = 0$ *RED* (b) $y(0) = 1$ *GREEN* (c) $y(0) = -1$ *BLUE*

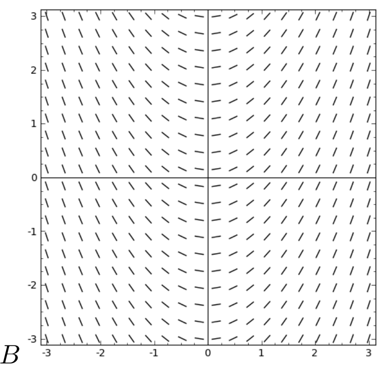


2
1
0
−1
−2
−2
−1
0
1
2

26. For each differential equation, find the corresponding slope field. (*Not all slope fields will be used.*)



A



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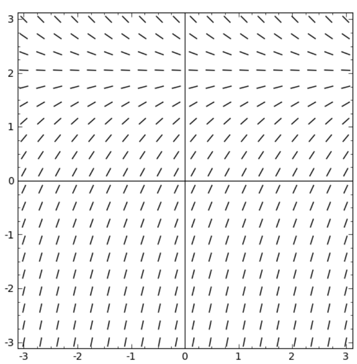

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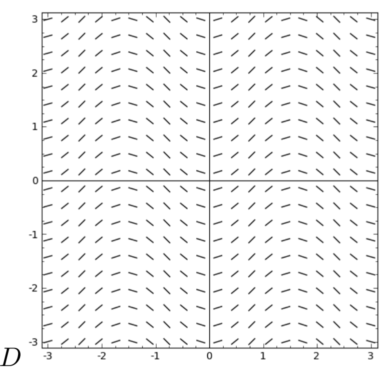

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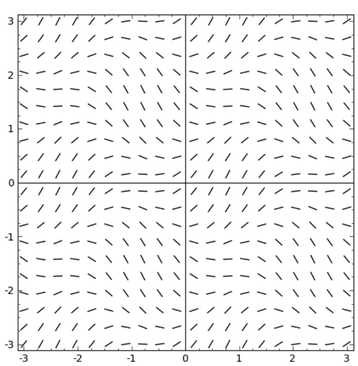

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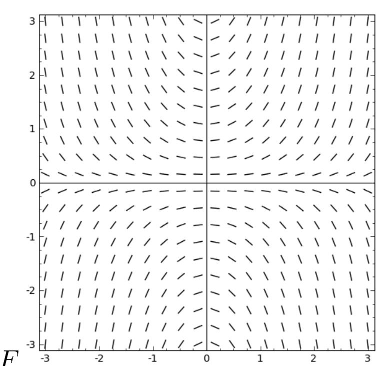


F

Equation	**Slope Field**
$\frac{dy}{dx} = x$	*B*
$y' = xy$	*F*
$y' = \sin(2x)$	*D*
$\frac{dy}{dx} = 2 - y$	*A*

27. Let $f(x, y) = e^{x^2y+xy^2}$.

(a) Find $f_x(x, y)$. $e^{x^2y+xy^2}(2xy + y^2)$

(b) Find $f_y(x, y)$. $e^{x^2y+xy^2}(x^2 + 2xy)$

(c) Find $\frac{\partial}{\partial x} f_y(x, y)$. $e^{x^2y+xy^2}(2x^3y + 5x^2y^2 + 2xy^3 + 2x + 2y)$

(d) Find $\frac{\partial}{\partial y} f_x(x, y)$. $e^{x^2y+xy^2}(2x^3y + 5x^2y^2 + 2xy^3 + 2x + 2y)$What's curious about the relation of this answer to that of part (c) above? They're equal.

28. Find the equation of the sphere that passes through the origin and whose center is $(1, 2, 3)$.

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = 14.$$

29. You want to estimate $\sin x$ using the first 3 nonzero terms in the Taylor series. What formula for the error bound would you use to get the best estimate for the error?

$$|E_6(x)| \le \frac{|x|^7}{7!}.$$

30. Find the arc length of the curve $y = x^{3/2}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.

$$\frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$$

31. Find the arc length of the curve $y = (x^6 + 8)/(16x^2)$ from $x = 2$ to $x = 3$. $\frac{595}{144}$

32. Let $f(x, y) = x^5 - 10x^3y^2 + 5xy^4$. Show that

$$\frac{\partial}{\partial x} f_x(x, y) + \frac{\partial}{\partial y} f_y(x, y) = 0.$$

$$\frac{\partial}{\partial x} f_x(x, y) = 20x^3 - 60xy^2; \quad \frac{\partial}{\partial y} f_y(x, y) = 60xy^2 - 20x^3.$$

33. Additional problems from your text:

   (a) Section 8.1 (Areas and Volumes) Exercises 3, 11, 25

3. By similar triangles, if w is the length of the strip at height h, we have

$$\frac{w}{3} = \frac{5-h}{5} \quad \text{so} \quad w = 3\left(1 - \frac{h}{5}\right).$$

Thus,

$$\text{Area of strip} \approx w\Delta h = 3\left(1 - \frac{h}{5}\right)\Delta h.$$

$$\text{Area of region} = \int_0^5 3\left(1 - \frac{h}{5}\right) dh = \left(3h - \frac{3h^2}{10}\right)\Bigg|_0^5 = \frac{15}{2}.$$

Check: This area can also be computed using the formula $\frac{1}{2}$ Base $\cdot$ Height $= \frac{1}{2} \cdot 3 \cdot 5 = \frac{15}{2}$.

11. Each slice is a circular disk. From Figure 8.2, we see that the radius at height y is $r = \frac{2}{5}y$ cm. Thus

$$\text{Volume of disk } \approx \pi r^2 \Delta y = \pi \left(\frac{2}{5}y\right)^2 \Delta y = \frac{4}{25}\pi y^2 \Delta y \text{ cm}^3.$$

Summing over all disks, we have

$$\text{Total volume } \approx \sum \frac{4\pi}{25} y^2 \Delta y \text{ cm}^3.$$

Taking the limit as $\Delta y \to 0$, we get

$$\text{Total volume } = \lim_{\Delta y \to 0} \sum \frac{4\pi}{25} y^2 \Delta y = \int_0^5 \frac{4\pi}{25} y^2 \, dy \text{ cm}^3.$$

Evaluating gives

$$\text{Total volume } = \frac{4\pi}{25} \frac{y^3}{3} \bigg|_0^5 = \frac{20}{3}\pi \text{ cm}^3.$$

Check: The volume of the cone can also be calculated using the formula $V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3} 2^2 \cdot 5 = \frac{20}{3}\pi$ cm^3.

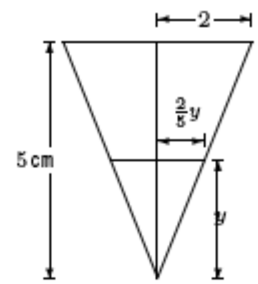
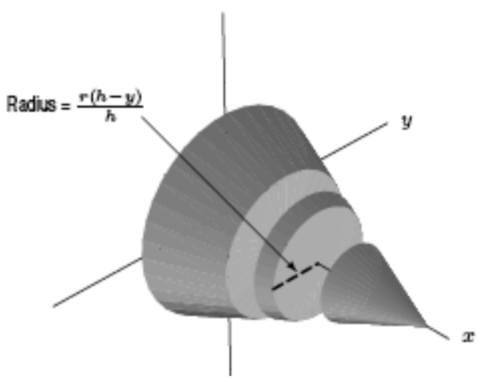


Figure 8.2

25.



This cone is what you get when you rotate the line $x = r(h-y)/h$ about the y–axis. So slicing perpendicular to the y–axis yields

$$V = \int_{y=0}^{y=h} \pi x^2 \, dy = \pi \int_0^h \left(\frac{(h-y)r}{h}\right)^2 dy$$

$$= \pi \frac{r^2}{h^2} \int_0^h (h^2 - 2hy + y^2) \, dy$$

$$= \frac{\pi r^2}{h^2} \left[h^2 y - hy^2 + \frac{y^3}{3}\right] \bigg|_0^h = \frac{\pi r^2 h}{3}.$$

(b) Section 8.2 (Volumes by Revolution & Cross Sections) Exercises 9, 23, 35

9. Since the graph of $y = x^2$ is below the graph of $y = x$ for $0 \le x \le 1$, the volume is given by

$$V = \int_0^1 \pi x^2 \, dx - \int_0^1 \pi (x^2)^2 \, dx = \pi \int_0^1 (x^2 - x^4) \, dx = \pi \left(\frac{x^3}{3} - \frac{x^5}{5}\right) \bigg|_0^1 = \frac{2\pi}{15}.$$

23. Note that the lines $x = 9$ and $y = \frac{1}{3}x$ intersect at $(9, 3)$. We slice the volume with planes that are perpendicular to the line $y = -2$. This divides the solid into thin washers with

$$\text{Volume of slice } = \pi r_{out}^2 \Delta x - \pi r_{in}^2 \Delta x.$$

Note that the inner radius is the vertical distance from the line $y = -2$ to the x-axis, so $r_{in} = 2$. Similarly, the outer radius is the vertical distance from the line $y = -2$ to the line $y = \frac{1}{3}x$, so $r_{out} = 2 + \frac{1}{3}x$. Integrating from $x = 0$ to $x = 9$ we have

$$V = \int_0^9 \left[\pi(2 + \frac{1}{3}x)^2 - \pi 2^2\right] dx.$$

35. Slicing perpendicularly to the x-axis gives squares whose thickness is Δx and whose side is $1 - y = 1 - x^2$. See Figure 8.26. Thus

$$\text{Volume of square slice } \approx (1 - x^2)^2 \Delta x = (1 - 2x^2 + x^4)\Delta x.$$

$$\text{Volume of solid } = \int_0^1 (1 - 2x^2 + x^4)\, dx = x - \frac{2}{3}x^3 + \frac{x^5}{5}\bigg|_0^1 = \frac{8}{15}.$$

(c) Section 8.3 (Arc Length and Parametric Curves) Exercises 11, 15, 17, 19

11. The graph will begin to draw over itself for any $\theta \geq 2\pi$ so the graph will look the same in all three cases. See Figure 8.41.

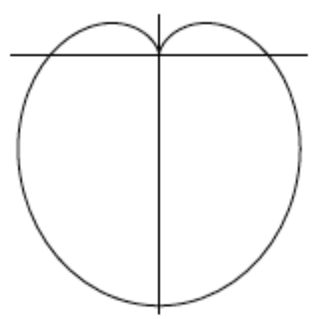


Figure 8.41

15. A loop starts and ends at the origin, that is, when $r = 0$. This happens first when $\theta = \pi/4$ and next when $\theta = 5\pi/4$. This can also be seen by using a trace mode on a calculator. Thus restricting θ so that $\pi/4 \le \theta \le 5\pi/4$ will graph the upper loop only. See Figure 8.47. To show only the other loop use $0 \le \theta \le \pi/4$ and $5\pi/4 \le \theta \le 2\pi$. See Figure 8.48.

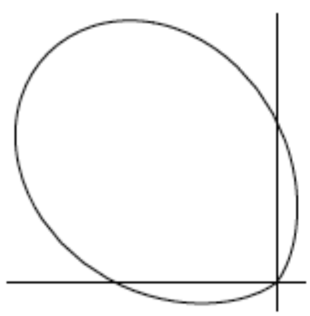


Figure 8.47: $\pi/4 \le \theta \le 5\pi/4$

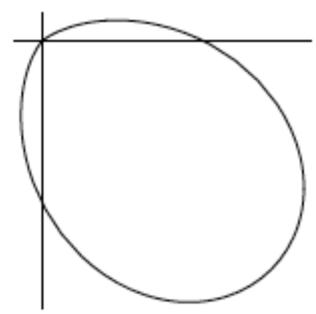


Figure 8.48: $0 \le \theta \le \pi/4$ and $5\pi/4 \le \theta \le 2\pi$

17. The region is given by $\sqrt{8} \le r \le \sqrt{18}$ and $\pi/4 \le \theta \le \pi/2$.

19. The circular arc has equation $r = 1$, for $0 \le \theta \le \pi/2$. the vertical line $x = 2$ has polar equation $r\cos\theta = 2$, or $r = 2/\cos\theta$. So the region is described by $0 \le \theta \le \pi/2$ and $1 \le r \le 2/\cos\theta$.

(d) Section 11.4 (Separation of Variables) Exercises 3, 9, 13, 21, 41

3. Separating variables and integrating both sides gives

$$\int \frac{1}{L} dL = \frac{1}{2} \int dp$$

or

$$\ln|L| = \frac{1}{2}p + C.$$

This can be written

$$L = \pm e^{(1/2)p + C} = Ae^{p/2}.$$

The initial condition $L(0) = 100$ gives $100 = A$, so

$$L = 100e^{p/2}.$$

9. Separating variables gives

$$\int \frac{dz}{z} = \int 5\,dt$$
$$\ln|z| = 5t + C.$$

Solving for z, we have

$$z = Ae^{5t}, \text{ where } A = \pm e^C.$$

Using the fact that $z(1) = 5$, we have $z(1) = Ae^5 = 5$, so $A = 5/e^5$. Therefore,

$$z = \frac{5}{e^5}e^{5t} = 5e^{5t-5}.$$

13. Separating variables gives

$$\int \frac{dy}{y-200} = \int 0.5dt$$
$$\ln|y-200| = 0.5t + C$$
$$y = 200 + Ae^{0.5t}, \quad \text{where } A = \pm e^C.$$

The initial condition, $y(0) = 50$, gives

$$50 = 200 + A, \quad \text{so} \quad A = -150.$$

Thus,

$$y = 200 - 150e^{0.5t}.$$

21. Separating variables gives

$$\frac{dz}{dt} = te^z$$
$$e^{-z}dz = tdt$$
$$\int e^{-z}\,dz = \int t\,dt,$$

so

$$-e^{-z} = \frac{t^2}{2} + C.$$

Since the solution passes through the origin, $z = 0$ when $t = 0$, we must have

$$-e^{-0} = \frac{0}{2} + C, \text{ so } C = -1.$$

Thus

$$-e^{-z} = \frac{t^2}{2} - 1,$$

or

$$z = -\ln\left(1 - \frac{t^2}{2}\right).$$

41. Separating variables gives

$$\frac{dx}{dt} = \frac{x \ln x}{t},$$

so

$$\int \frac{dx}{x \ln x} = \int \frac{dt}{t},$$

and thus

$$\ln |\ln x| = \ln t + C,$$

so

$$|\ln x| = e^C e^{\ln t} = e^C t.$$

Therefore

$$\ln x = At, \quad \text{where } A = \pm e^C \quad \text{or} \quad A = 0, \quad \text{so} \quad x = e^{At}.$$