6360 HW

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Abstract. This is a running list of homework assigned in class.

1. January

Exercise 1.1. Prove the chain rule: Suppose

$$f : \mathbb{R}^n \to \mathbb{R}^m \quad \text{and} \quad g : \mathbb{R}^m \to \mathbb{R}^\ell$$

are differentiable at p and $f(p)$ respectively. Then show that $g \circ f$ is differentiable at p, and $D(g \circ f)_p = Dg_{f(p)} \circ Df_p$.

Exercise 1.2. Consider the function $q : \mathbb{R} \to \mathbb{R}$ defined by

$$q(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}, \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Define $f(x) = x^2 q(x)$. Show that $f'(0) = 0$, but that $f(x)$ is not even continuous (let alone differentiable) at any $x \neq 0$.

Exercise 1.3. Consider the function $s : \mathbb{R} \to \mathbb{R}$ defined by

$$s(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin(1/x) & \text{if } x \neq 0. \end{cases}$$

Define $f(x) = x^2 s(x)$. Show that $f'(0) = 0$, that $f(x)$ is differentiable at each $x \in \mathbb{R}$, but that $f'(x)$ is not continuous at 0.

Exercise 1.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Show that f is differentiable at each point other than the origin. Show that

$$\partial f/\partial x(0, 0) = \partial f/\partial y(0, 0) = 0.$$

Thus the partials $\partial f/\partial x$ and $\partial f/\partial y$ exist on $\mathbb{R}^2$. Show, however, that $f(x, x) = 1$ for all $x \neq 0$, so that f is not continuous at $(0, 0)$ (let alone differentiable).

Let $p \in \mathbb{R}^n$. Consider the set

$$S := \{(f, U)_p : p \in U \subseteq \mathbb{R}^n, f : U \to \mathbb{R}\}.$$

Define an equivalence relation on the set by the rule

$$(f, U)_p \sim (g, V)_p$$

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if there exists an open neighborhood of p, $W \subseteq U \cap V$ such that $f|_W = g_W$. We then define the set of germs of functions at p to be the quotient S by this equivalence relation.

$$\mathrm{Map}(\mathbb{R}^n, \mathbb{R})_p := S/\sim .$$

We refer to the elements as germs of functions on $\mathbb{R}^n$ at p.

Exercise 1.5. Show that $\sim$ is an equivalence relation. For $(f, U)_p$ and $(g, V)_p$ germs of functions, define

$$(f, U)_p + (g, V)_p = (f|_{U\cap V} + g|_{U\cap V}, U \cap V)_p.$$

Define multiplication of germs similarly. Show that this gives the set of germs a ring structure (you must first show that this gives a well defined addition map for germs, etc.). Define a map $\mathbb{R} \to \mathrm{Map}(\mathbb{R}^n, \mathbb{R})_p$ by $r \mapsto (r, \mathbb{R}^n)_p$. Show that this is a homomorphism of rings, and so this defines an $\mathbb{R}$-algebra structure on the ring.

Exercise 1.6. For an open set $U \subseteq \mathbb{R}^n$, we denote by $C^\infty(U)$ the set of real valued smooth functions on U. That is, the functions that have continuous partial derivatives of all order. Define the set of germs of smooth functions at a point $p \in \mathbb{R}^n$ similarly. Show that this has a similar structure as an $\mathbb{R}$-algebra. We will denote this by $C^\infty(U)_p$.

We denote the set of all A-derivations of B into M by $\mathrm{Der}_A(B, M)$.

Exercise 1.7. Let A be a commutative ring with unity, let B be an A-algebra, and let M be a B-module. Show that $\mathrm{Der}_A(B, M)$ has a natural structure as an A-module. (I.e. $(D_1 + D_2)(b) := D_1(b) + D_2(b)$, etc.)

Exercise 1.8. If $f : U \to \mathbb{R}^m$ is a C^∞ map, with $U \subseteq \mathbb{R}^n$ an open subset, and $p \in U$, then show there is a linear map

$$T_p f : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$$

given by

$$D \mapsto T_p f(D)$$

where

$$T_p f(D)\left[(g, V)_{f(p)}\right] := D\left[(g \circ f, f^{-1}(V))_p\right] \in \mathbb{R}.$$

Exercise 1.9. For each $i = 1, \ldots, n$ we have a map $\partial/\partial x_i$ defined by the rule that for a smooth germ $(f, U)_p$,

$$\frac{\partial}{\partial x_i}(f, U)_p := \frac{\partial f}{\partial x_i}(p).$$

Show that $\partial/\partial x_i$ is a derivation.

Exercise 1.10. From what we have shown in class, we have a diagram

$$\begin{array}{ccc} T_p\mathbb{R}^n & \xrightarrow{T_p f} & T_p\mathbb{R}^m \\ \downarrow & & \downarrow \\ \mathbb{R}^n & \xrightarrow{Df_p} & \mathbb{R}^m, \end{array}$$

where the vertical arrows are isomorphisms induced by a choice of co-ordinates. Show that the diagram is commutative. [Hint: use the Jacobian matrix.]

Exercise 1.11. Show that the group of two by two real conformal matrices can be described as:

$$CO(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : (a,b) \in \mathbb{R}^2 - \{0\} \right\}.$$

Recall the statement of the implicit function theorem.

Theorem 1.12 (Implicit function theorem)**.** *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \to \mathbb{R}^m$ be in $C^r(U, \mathbb{R}^m)$, where $n \geq m$. Fix a point $p \in f^{-1}(0)$. Consider the "vertical" affine m space V passing through p; more precisely set:*

$$V := \{p' \in \mathbb{R}^n : x_1(p') = p_1, \dots, x_{n-m}(p') = p_{n-m}\}.$$

If

$$T_pV \cap \ker Df_p = 0,$$

then there exists a neighborhood $U' \subseteq U$ of p, a neighborhood W of $(p_1, \dots, p_{n-m})$ in $\mathbb{R}^{n-m}$ and a $C^r(W, \mathbb{R}^m)$ map g such that

$$f^{-1}(0) \cap U' = \Gamma_g := \{(x, g(x) : x \in W\}.$$

In other words, $f^{-1}(0)$ is locally the graph of a C^r function in the first $n-m$ coordinates.

Exercise 1.13. Suppose that $\dim\ker Df_p = n - m$ and $T_pV \cap \ker Df_p \neq 0$. Show that there does not exits a neighborhood W of $(p_1, \dots, p_{n-m})$ and a smooth function $g : W \to \mathbb{R}^n$ such that $f^{-1}(0)$ is locally the graph of g. [Hint: First consider the composition $f \circ (Id \times g)$. Then consider a path $\gamma : (a,b) \to \mathbb{R}^n$ through p, lying on the graph Γ_g, with non-trivial tangent vector. Show that there is some $i \in 1, \dots, n-m$ such that $\gamma_i'(t) \neq 0$. In other words, $\gamma'(t) \notin T_pV$.]

Exercise 1.14. Consider the example with $f(x,y) = x - y^3$. Show that $T_pV \cap \ker Df_p \neq 0$, but there exits a neighborhood W of the origin in $\mathbb{R}$ and a function $g : W \to \mathbb{R}$ such that $f^{-1}(0)$ is locally the graph of g.

Exercise 1.15. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ morphism such that Df_p is surjective for some $p \in f^{-1}(0)$. Then there is a neighborhood U of p such that $f^{-1}(0) \cap U$ is the graph of a C^∞ function (not necessarily in the first $n-m$ coordinates). Note the condition that Df_p be surjective can be replaced with the condition that $\dim\ker(Df_p) = n - m$. [Hint: consider precomposing f with a linear isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$.]

Exercise 1.16. Show that there exist functions f satisfying all the conditions of the theorem, except that $T_pV \cap \ker Df_p \neq 0$, but with the property that $f^{-1}(0)$ is still locally the graph of a function in the first $n - r$ coordinates. [Hint: consider the function $f(x,y) = (x-y)^2$. Note that in this example $\dim\ker Df_p > n - m$.]

Exercise 1.17. If you know the definition of a manifold: show that the function $f(x,y) = y^2 - x^3$ satisfies all of the conditions of the theorem, except that we have $\dim\ker Df_0 > n-m = 1$ (and of course $T_pV \cap \ker Df_p \neq 0$). Show that $f^{-1}(0)$ is not a sub-manifold of $\mathbb{R}^2$.

Exercise 1.18. Show that there is an isomorphism of rings

$$\phi : \mathbb{C} \to \widehat{CO}(2, \mathbb{R})$$

given by

$$a+ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Recall that $\widehat{CO}(2,\mathbb{R})$ is the union of the conformal matrices with the zero matrix, and addition and multiplication on this ring are given in terms of matrix addition and multiplication.

Exercise 1.19. Consider a linear map

$$A : \mathbb{R}^2 \to \mathbb{R}^2.$$

We then get a diagram

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \| & & \| \\ \mathbb{C} & & \mathbb{C}. \end{array}$$

Show that there exists a linear map $\alpha \in M(1,\mathbb{C}) = \mathbb{C}$ making the above diagram commute if and only if $A \in \widehat{CO}(2,\mathbb{R})$.

Exercise 1.20. Identifying $\mathbb{C}$ with $\mathbb{R}^2$, a C^∞ function $f_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $\zeta = a+ib$ if and only if $(Df_{\mathbb{R}})_{(a,b)} \in \widehat{CO}(2,\mathbb{R})$.

Exercise 1.21. Let $f : U \to \mathbb{C}$ be a holomorphic map. Suppose that $f'(\zeta) \neq 0$ for some $\zeta \in U$. Then f is a local holomorphic isomorphism near ζ. More precisely, there exists an open neighborhood $U' \subseteq U$ of ζ such that $V := f(U')$ is an open neighborhood of $f(\zeta)$, $f|_{U'}$ is one-to-one, and the set map f^{-1} is in fact holomorphic on V.

Exercise 1.22. Consider a function $f : U \to \mathbb{C}$. Write

$$f(x+iy) = u(x,y) + iv(x,y)$$

for some u, v that are real valued functions of real numbers. Suppose that $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}$ exist and are continuous in a neighborhood V of (a,b). Show f is $\mathbb{C}$-differentiable at $\zeta = a+ib$ if and only if the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

hold at (a,b). Consequently, f is holomorphic in V if and only if the Cauchy-Riemann equations hold at each point of V.

Exercise 1.23. Find a function $f(z)$ that is holomorphic on a connected open set U, such that there does not exist a function $F(z)$ holomorphic on U such that $F'(z) = f(z)$. [Hint: consider the domain of definition of $\log z$.]

Exercise 1.24. Let S be a subset of a topological space X. Show that the closure of S (the intersection of all closed subsets containing S) is equal to the set of points $p \in X$ such that for every open neighborhood U of p, $U \cap X \neq \emptyset$.

Exercise 1.25. Recall that a limit point (or accumulation point) p of a subset S of a topological space X is a point $p \in X$ such that for every open set U containing p, there exists $q \in U \cap S$ with $q \neq p$. Show the closure of a set S is the disjoint union of S with those limit points not in S. Thus a set is closed if and only if it contains all of its limit points.

Exercise 1.26. The set of limit points of a closed set is closed.

Exercise 1.27. Find a topological space X and a subset $S \subseteq X$ such that the set of limit points of S is not closed.

Exercise 1.28. Let X be a T_1 topological space. Show that the limit set of any subset of X is closed.

2. February

Exercise 2.1. Suppose that U is a simply connected subset of $\mathbb{R}^2$, and $u \in C^2(U)$ is harmonic. Then we showed there exists a harmonic function $v \in C^2(U)$ such that

$$f(z) := u + iv$$

is holomorphic. In particular, $u \in C^\infty(U)$.

Show that the statement is false without the assumption that U be simply connected. [Hint: consider $u = \ln|z|$. Then if $\ln|z| + iv$ is analytic, then $v(z) = \operatorname{Arg} z + a$ except along the non-positive real axis.]

Exercise 2.2. Suppose that $u \in \mathcal{H}(U)$ is a harmonic function. For $a \in U$, and

$$p \in V(u-a) := \{(x, y) \in U : u(x, y) - a = 0\}$$

show that if $T_p u \neq 0$, then $V(u-a)$ is a smooth curve in the plane near p. That is to say, there exists an open neighborhood U' of p in U, an open interval $0 \in (a, b) \subseteq \mathbb{R}$ and a C^∞ map

$$\gamma : (a, b) \to \mathbb{R}^2$$

such that $\gamma(0) = p$ and $V(u-a) \cap U' = \gamma((a, b))$. [Hint: use the implicit function theorem]

Exercise 2.3. In the notation of the previous exercise, assume that U is simply connected, let v be the harmonic conjugate of u, and assume $p \in V(v)$ and $p \in V(u)$. Let δ be the C^∞ map defining the smooth zero set of v near p. Show that $Image(T_0\gamma) \perp Image(T_0\delta)$. In other words, the level sets of the harmonic conjugate are orthogonal to the level sets of the harmonic function.

Exercise 2.4. Use the previous problem to show that if u is harmonic on a simply connected open set, and F is the vector field given by the differential of u (that is the gradient vector field), then the harmonic conjugate v of u has level sets that are parallel to the vector field F.

Exercise 2.5. Show that the Poisson kernel

$$P(r, t) := \sum_{n \in \mathbb{Z}} r^{|n|} e^{int} = \lim_{N \to \infty} \sum_{n=-N}^{N} r^{|n|} e^{int}$$

converges (abosolutely) uniformly for all $0 \leq r < 1$ and satisfies the following properties.

(1) For each $\theta \in C$, we have

$$P(r, \theta - t) = \operatorname{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] = \frac{(1 - r^2)}{1 - 2r\cos(\theta - t) + r^2}.$$

Note that it follows that $P(r, t) \geq 0$.

(2) For f continuous on C, and setting $\zeta = re^{i\theta}$, we have

$$u_f(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t) f(e^{it}) dt.$$

(3) For a trigonometric polynomial g, $\bar{u}_g \in C(\bar{B})$.

(4) We have

$$\frac{1}{2\pi}\int_0^{2\pi} P(r,\theta - t) dt = 1.$$

Exercise 2.6. Let $f : U \to \mathbb{C}$ be a smooth map. Then there is an induced map

$$T_p f : T_p^{\mathbb{C}} U \to T_{f(p)}^{\mathbb{C}} \mathbb{C}.$$

Show that f is holomorphic if and only if this map takes T^h to T^h.

Exercise 2.7. Let $f : U \to V$ be a continuous map of open subsets of the complex plane. Show that f is holomorphic if and only if for every $g \in \mathscr{O}(V)$, $f^*g \in \mathscr{O}(U)$.

Exercise 2.8. Show that the fractional linear transformations form a group under composition isomorphic to $\mathbb{P}GL_2(\mathbb{C})$.

Exercise 2.9. Show that any fractional linear transformation can be decomposed into the composition of maps of the form $f(z) = z + b$, $f(z) = az$ and $f(z) = 1/z$.

Exercise 2.10. Show that the maps $f(z) = z + b$ and $f(z) = az$ send lines to lines, and circles to circles. Show that the map $f(z) = 1/z$ sends a line to either a line or a circle (depending on whether the line passes through zero), and sends a circle to either a line or circle (depending on whether the circle passes through zero). Conclude that fractional linear transformations send circles and lines to circles and lines.

Exercise 2.11 (Turn in). Let $\pi_N : (\Sigma - \{N\}) \to \mathbb{R}^2$ and $\pi_S : (\Sigma - \{S\}) \to \mathbb{R}^2$ be the projections from the north and south poles respectively of the unit sphere Σ.

Let $U_1 = U_2 = \mathbb{C}$, and define map

$$\phi_1 : U_1 \to (\Sigma - \{N\})$$

by $\phi_1(z_1) = \pi_N^{-1}(z_1)$ (where we have identified $\mathbb{C}$ with $\mathbb{R}^2$). Define a map

$$\phi_2 : U_2 \to (\Sigma - \{S\})$$

by $\phi_2(z_2) = \pi_S^{-1}(\bar{z}_2)$. Show that the composition

$$(U_1 - \{0\}) \xrightarrow{\phi_1} \Sigma - \{N, S\} \xrightarrow{\phi_2^{-1}} (U_2 - \{0\})$$

is given by $z_2 = 1/z_1$.

Exercise 2.12. Find a series centered at 0 that has radius of convergence 1, but does not converge at any point of $|z| = 1$. Find a series centered at 0 that has radius of convergence 1 and does converge at all $|z| = 1$.

Exercise 2.13. Using taylor's theorem applied to a branch of $\ln(1 + z/n)$ prove that

$$\lim_{n\to\infty}\left(1 + \frac{z}{n}\right)^n = e^z$$

uniformly on compact sets.

Exercise 2.14. Show that the series

$$\zeta(s) = \sum_{i=1}^{\infty} n^{-s}$$

converges uniformly on compact sets with $\operatorname{Re} s > 1$ and represent its derivative in series form.

Exercise 2.15. Show that the Laurent development of a function is unique.

Exercise 2.16. Find the Laurent series expansion for $1/(z-1)(z-2)$ in the region $1 < |z| < 2$.

Exercise 2.17. Show that the series

$$\sum_{n \neq 0} \frac{z}{n(z-n)}$$

converges absolutely uniformly on compact subsets of $\mathbb{C} - \mathbb{Z}$.

Exercise 2.18 (Turn in). Show that the Laurent development for $(e^z - 1)^{-1}$ at the origin is of the form

$$\frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$$

for some numbers B_k. These are the Bernoulli numbers. Calculate B_1, B_2, B_3.

Exercise 2.19 (Turn in). Express the Taylor development of $\tan z$ and the Laurent development of $\cot z$ in terms of the Bernoulli numbers. [Hint: it may be useful to use the relation $\tan z = \cot z - 2 \cot 2z$.]

Exercise 2.20 (Turn in). Comparing coefficients in the Laurent developments of $\cot \pi z$ and of its expression as a sum of partial fractions, find the values of $\zeta(2)$, $\zeta(4)$, $\zeta(6)$.

Exercise 2.21 (Turn in). More generally, show that

$$\zeta(2k) = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k}.$$

Exercise 2.22. Show that

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

Exercise 2.23. Show that if A is a subset of $\mathbb{R}^n$ with no limit points, then it is countable.

Exercise 2.24. Let Z be the zero set of a non-zero entire function f. Show that if Z is infinite there is an enumeration $a_1, \ldots$ of the points of Z such that $\lim_{n\to\infty} a_k = \infty$.

Exercise 2.25. Show that

$$\prod_{k=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

converges uniformly on compact subsets of $\mathbb{C}$.

Exercise 2.26. Suppose that $a_n \to \infty$ and that A_n are arbitrary complex numbers. Show that there exists an entire function $f(z)$ which satisfies $f(a_n) = A_n$.

Exercise 2.27. Let S be a set, and let $\{f_k : S \to \mathbb{R}\}$ be a sequence of bounded functions such that

$$\sum_{k=1}^{\infty} f_k(s)$$

converges uniformly on S. Let $f : S \to \mathbb{R}$ be the limit function. Show that f is bounded.

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