# Sets versus Classes: Why you care.

Bart Kastermans

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**Abstract**

Notes for a talk in the Stacks and Groupoids seminar at the Mathematics Department at the University of Colorado. Things are kept relatively informal.

## 1 Introduction

We want to talk of the categories $\mathfrak{Top}$ (Topological Spaces), $\mathfrak{Rings}$ (Rings), $\mathfrak{RSpaces}$ (Ringed Spaces), and of Grothendieck Topologies. Depending on what you want with them some foundations comes into the picture. We give in this talk a hint of what these foundations are and how they work.

The key idea is the notion of a universe. In Section 3 we give the idea. Then we show in Section 4 that they don't really exist. In Section 5 we show some other important objects that don't exist. In Section 6 we explain in which way some of the earlier objects can be used anyway. This use is somewhat restrictive, so in Section 7 we introduce the assumption that one or more universes exist. Finally in Section 8 we indicate how it might work in practice.

## 2 The First Hint

Sets for a long time were only used intuitively to mean collections of objects. Working with them as just collections of objects does not mean you don't need to come up with proper ways of reasoning about them. A common construction was to collect elements together:

$$\{x \mid \varphi(x)\}$$

The set of all objects satisfying the property $\varphi$. From this you get the Russell paradox:

$$R = \{x \mid x \notin x\},$$

with the accompanying question: is $R$ a member of $R$?

# 3 What is a Universe?

There are many different solutions to this problem; that is many different Set Theories that attempt to solve this problem. Here we won't really be using any of them, but always have the "usual" one in mind, that is $\mathsf{ZFC}$.

On wikipedia (http://en.wikipedia.org/wiki/ZermeloFraenkel_set_theory) you can find all the axioms. The idea is that we do not define what a set is, but what properties the universe of sets should satisfy.

Then a **universe of sets** is any collection of objects ($\mathsf{V}$) with a binary relation ($\in$) such that the collection satisfies the axioms. This is exactly analogous to saying that a group is a collection of objects with a binary rule that satisfies the axioms of group theory.

The axioms are intuitively:

- two sets are equal if they have the same elements,
- if $a$ and $b$ are sets, then so is $\{a, b\}$,
- if $a$ is a set, then $\mathcal{P}(a)$ is a set,
- if $a$ is a set, then so is $\bigcup a$,
- the image of a set under a function is a set,
- things of the form $x \in x$ and more general don't happen,
- a set containing all natural numbers exists,
- every set can be wellordered,
- **(Comprehension) if $a$ is a set, then $\{x \in a \mid \varphi(x)\}$ is a set.**

When using this as a foundation of mathematics, then *everything* is a set. All groups, spaces, manifolds, functions are sets. This means something exists if it is a set. Our axioms are strong enough to get representatives for all of the objects you would want in the ordinary working of mathematics, but it is not boundless (as witnessed by the contradiction obtained from $R$ in the previous section).

Also a universe is closed under any construction you can do. You can imagine a construction as a function $F : \mathsf{V}^n \to V$ taking some inputs and giving as its output the result of the constrution. Showing that you can always perform the constrution of an object $y$ from objects $x_0, \ldots, x_n$ exactly amounts to showing $F$ has the type as indicated.

# 4 Existence of Universes

We cannot show a universe of sets exists, since this would imply the consistency of $\mathsf{ZFC}$. Since we believe we "live and work" in $\mathsf{ZFC}$ this would mean $\mathsf{ZFC}$ proves its own consistency, which since Gödel we know is impossible.

However since we believe we "live and work" in $\mathsf{ZFC}$ the collection of all objects would be a universe if only it existed:

$$\mathsf{V} = \{x \mid x = x\}$$

We can prove it does not exist since if $\mathsf{V}$ is a set, then by comprehension so would be $\{x \in \mathsf{V} \mid x \notin x\}$ which is the Russel set.

## 4.1 Ordinals and Cardinals

For a finite set $\{a, b, c\}$ we can say it has three elements: three is a mathematical object uniquely denoting the size of this set. We want to do this for infinite sets too, where two sets are of the same size iff there is a bijection between them. The smoothest way to do this seems to be through ordinals.

Ordinals are uniquely representing the size of wellordered sets (a wellorder is a linear order in which every nonempty set has a least element). We can define $\mathsf{Ord} = \{\alpha \mid \alpha \text{ is an ordinal } \}$. The class of ordinals $\mathsf{Ord}$ is itself an ordinal in every way except that it is not a set (it is a wellordered class).

Then defining a cardinal as an initial ordinal (an ordinal that is not in bijection with anything smaller) we get a notion of cardinal that satisfies the requirement stated in the above paragraph: every infinite set is uniquely represented. We can define $\mathsf{Card} = \{\kappa \mid \kappa \text{ is a cardinal }\}$. Again the class of cardinals $\mathsf{Card}$ is wellordered (which is the property that allows transfinite induction on the sizes of sets).

## 4.2 Picture of the Universe

We define the cumulative hierarchy:

$$\begin{aligned} V_0 &= \emptyset \\ V_{\alpha+1} &= \mathcal{P}(V_\alpha) \\ V_\lambda &= \bigcup_{\alpha<\lambda} V_\alpha (\text{when } \lambda \neq \alpha+1) \end{aligned}$$

It can be shown that $\mathsf{V} = \bigcup_{\alpha\in\mathsf{Ord}} V_\alpha$ (i.e. every set is in $V_\alpha$ for some $\alpha$). From this it follows that $\mathsf{Ord}$ is not a set.

# 5 If Too BIG, then not a Set

The general intuitive rule is that something that is too big is not a set. Check how the axioms deal with this.

We have seen $\mathsf{V}$ and $\mathsf{Ord}$ are not sets. The usual way to show that other things are not sets is to show that if they were then something like $\mathsf{V}$ or $\mathsf{Ord}$ is also a set.

## 5.1 One Point Topological Spaces

Consider the collection of one point topological spaces $\mathsf{ONE}$. Define:

$$\mathsf{ONE}_x = (\{x\}, \{\{x\}, \emptyset\})$$

This in essence gives a map $\mathsf{V} \to \mathsf{ONE}$ that is a bijection, therefore so is its inverse. If now $\mathsf{ONE}$ were a set, then so would be the image under the map, but this image is $\mathsf{V}$ which is not a set.

## 5.2 Almost all other Categories

Let $C$ be any category, and $z$ an object in $C$. For many categories $z$ has some carier set, where the exact identity of the elements in this set do not really matter. Pick one such point to function as $x$ in the story on one point topological spaces, and a similar argument works.

This certainly works for $\mathfrak{Top}$ and $\mathfrak{Rings}$.

### 5.3 Grothendieck Topologies

Lets use the definition of Grothendieck Topology applied to the category $\mathfrak{Top}$. Such a Grothendieck Topology assigns to every object $X \in \mathfrak{Top}$ a collection of sieves $J(X)$. $J(X)$ is always not empty, since it always contains the maximal sieve. Note that from any sieve $S \in J(X)$ we can recover $X$ (the codomain of any arrow in the sieve).

Now if the topology $J$ were a set, then so would its range $R = \bigcup_{X \in \mathfrak{Top}} \{J(X)\}$. From this range by the last observation in the last paragraph we can then recover $\mathfrak{Top}$ as a set; contradiction.

The conclusion here is that Grothendieck Topologies on such categories never exist, but there might be classes that are Grothendieck Topologies. In that case you however certainly cannot effectively work with the collection of all Grothendieck Topologies.

### 5.4 $\mathfrak{Sets}^{\mathfrak{C}^{\mathrm{op}}}$

If $\mathfrak{C}$ is any category we would like to consider $\mathfrak{Sets}^{\mathfrak{C}^{\mathrm{op}}}$, the functor category. If $\mathfrak{C}$ is not a set, but only a class, we cannot do this. Any function $F : \mathfrak{C} \to \mathfrak{Sets}$, if it exists, then so does its domain, which does not exist. So all such $F$ would be classes, and thus cannot be a member of anything, in particular not a collection of all such functors.

## 6 Classes don't Exist, but are Useful

We introduced several notations so far for things that don't exist: $\mathsf{V}$, $\mathfrak{Top}$, $\mathsf{Ord}$, $\mathfrak{Rings}$. These are still very useful though, if you are careful you can use them effectively.

$\mathfrak{Top}$ is the class of topological spaces with continuous maps. So it really consists of two formulas $\varphi_O$ and $\varphi_A$ such that $\varphi_O$ is a formula true exactly of the objects for the category, and $\varphi_A$ is true exactly of the arrows of the category. This is generally the case for classes, they are identified with formulas. Using $\mathfrak{Top}$ as an abbreviation makes it very easy to write: for all $x \in \mathfrak{Top}$

something happens. This is then an abbreviation of for all $x$ such that $\varphi_O(x)$ something happens.

What a class can not do however is be an element of something. And you can't take a powerclass and have something that behaves like you might hope (the powerclass of $\mathfrak{Top}$ should contain $\mathsf{ONE}$, but $\mathsf{ONE}$ is not a set so can't be in anything).

## 6.1 For Categories

We now can also distinguish between different types of categories. There are the *concrete categories*, those are categories that exist as sets, i.e. a set of objects and a set of arrows. And there are class categories, categories where either the collection of objects, or the collection of arrows do not form a set. For the class categories you have to be more careful with your constructions, i.e. see the collection of Grothendieck Topologies.

# 7 A Solution: Universes

$\mathsf{V}$ is where we can imagine all mathematics to take place **because** it satisfies the axioms. Suppose we had a $U \in \mathsf{V}$ such that $U$ also satisfies all the axioms. Then $U$ could be the universe of ordinary mathematics, and we could be working in $\mathsf{V}$ to study $U$. To simplify the picture we can assume $U = V_\kappa$ for some $\kappa \in \mathsf{Card}$. This also gives us both a nice picture and an easy way to extend in case we need more universes. We could assume we have $U_0 \in U_1 \in U_2 \in \mathsf{V}$ all universes, and all different levels of the cumulative hierarchy.

So lets assume we have a universe $V_\kappa$. Then we can, in stead of studying $\mathfrak{Top}$, study those topological spaces and maps that are in $V_\kappa$

$$\mathfrak{Top}_\kappa := \{X \in V_\kappa \mid X \in \mathfrak{Top}\}$$

analogously

$$\mathfrak{Rings}_\kappa := \{R \in V_\kappa \mid R \in \mathfrak{Rings}\}$$

Now because $V_\kappa$ is a universe whatever constructions you do in $\mathfrak{Top}_\kappa$ and $\mathfrak{Rings}_\kappa$ you still remain in $V_\kappa$ (any construction, see the last paragraph of Section 3, $F : \mathsf{V}^n \to \mathsf{V}$ that can be shown to work in $\mathsf{ZFC}$ has the property that if $U$ is a universe, then $F \restriction U : U^n \to U$ for the same reason given in that last paragraph of Section 3).

A common way of speaking when you have a universe $V_\kappa$ is to call the elements of $V_\kappa$ *small* sets. This means that $\mathfrak{Top}_\kappa$ is the collection of small topological spaces. Note that $\mathfrak{Top}_\kappa \subset V_\kappa$, but $\mathfrak{Top}_\kappa \notin V_\kappa$ since $\mathfrak{Top}_\kappa$ is too large (an argument analogous to the one that $\mathfrak{Top}$ is not a set can be used here with the fact $V_\kappa \notin V_\kappa$).

When you have multiple universes $V_{\kappa_0} \in V_{\kappa_1}$ you might call the elements of $V_{\kappa_0}$ the *very small sets* and the elements of $V_{\kappa_1}$ the small sets. Note that $V_{\kappa_0}$ is a small set, but not a very small set. Since $V_{\kappa_0}$ is a small set, from it we can define $\mathfrak{Rings}_{\kappa_0}$ the category of very small rings. This being defined from $V_{\kappa_0}$ the category of very small rings, is represented by a small set (but not a very small set).

# 8 In Practice

Lets start by looking at ringed spaces ($\mathfrak{RSpaces}$), these are functors with domain a topological space, and range in $\mathfrak{Rings}$.

Then we can define $\mathfrak{RSpaces}_\kappa$, the category of all ringed spaces in $V_\kappa$. We can alternatively use elements of $\mathfrak{Top}_\kappa$ as domain, and $\mathfrak{Rings}_\kappa$ as codomain; i.e. functors $F : X \to \mathfrak{Rings}_\kappa$ with $X \in \mathfrak{Top}_\kappa$. Two very different looking strategies.

Since $V_\kappa$ is a universe, the axioms hold in it. This means that $F[X] \in V_\kappa$ (the image of a set is a set). Then $F \subseteq X \times F[X]$ and $X \times F[X] \in V_\kappa$. Then also $\mathcal{P}(X \times F[X]) \in V_\kappa$, which contains $F$, and therefore $F \in V_\kappa$.

On the other hand any ringed space in $V_\kappa$ certainly is a functor $F : X \to \mathfrak{Rings}_\kappa$. So both approaches coincide after all (which certainly is a comfort).

Now if $\mathfrak{C} \in V_\kappa$ we can also form $(V_\kappa)^{\mathfrak{C}^{\mathrm{op}}}$, which will be a concrete category, the analogue of the not concrete category $\mathfrak{Sets}^{\mathfrak{C}^{\mathrm{op}}}$.

We can now also form the collection of Grothendieck Topologies on a category like $\mathfrak{Top}_\kappa$, since the collection of all maps in $\mathfrak{Top}_\kappa$ is a set (although not a small set), and Grothendieck Topologies are subsets of this collection.