Generic vanishing results on certain Koszul cohomology groups

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1 Introduction

1.1 Background

A central problem in curve theory is to study to extrinsic geometry of curves in projective spaces with fixed genus and degree. Koszul cohomology groups in some sense carry 'everything one wants to know' about the extrinsic geometry of curves in projective space: the number of equations of each degree needed to define the curve, the relations between the equations, etc. In this poster, I will present a new method using deformation theory to study Koszul cohomology of general curves.

Let X be a projective variety and L be a globally generated line bundle with $r+1$ sections on X. The information of the geometry of the map $\phi_{|L|} : X \longrightarrow \mathbb{P}^r$ is encoded in the section ring

$$R = R(X, L) = \bigoplus_{k\geq 0} H^0(X, L^k).$$

Let $V = H^0(X, L)$ and form the symmetric algebra $S = \bigoplus_{k\geq 0} S^k V$. R is a graded module over S and admits a minimal free resolution of graded S-modules

$$\cdots \to \bigoplus_{q\geq 0} S(-p-q)^{k_{p,q}} \to \cdots \to \bigoplus_{q\geq 0} S(-1-q)^{k_{1,q}} \to \bigoplus_{q\geq 0} S(-q)^{k_{0,q}} \to R \to 0.$$

We can spell out the geometric information from the resolution. For example, L is normally generated if and only if $k_{0,q} = 0$ for $q \geq 1$ and if this is the case, then

$$k_{1,q} = \text{\# of degree } (q+1) \text{ primitive generators of } I_X.$$

Although the resolution is not unique, the free S-module at each step p is uniquely determined by R:

$$S(-p-q)^{k_{p,q}} = K_{p,q}(X, L) \otimes S(-p-q),$$

where $K_{p,q}(X, L)$ is the span of minimal generators of degree $p+q$ in step p and can be computed via the cohomology of the Koszul complex at (p, q)-spot:

$$\to \wedge^{p+1} V \otimes H^0(X, L^{q-1}) \to \wedge^p V \otimes H^0(X, L^q) \xrightarrow{d_{p,q}} \wedge^{p-1} V \otimes H^0(X, L^{q+1}) \to$$

where

$$d_{p,q}(v_1 \wedge \ldots \wedge v_p \otimes \sigma) = \sum_i (-1)^i v_1 \wedge \ldots \wedge \widehat{v}_i \wedge .. \wedge v_p \otimes v_i \sigma.$$

We are interested in Green's question:

Problem 1. *(Green) What is the variational theory of the $K_{p,q}(X, L)$? What do they look like for X a general curve and L a general g^r_d?*

1.2 What is known/unknown

If (X, L) is general in $\mathcal{G}^r_{g,d}$ (here it means (X, L) is a general point of the unique component of $\mathcal{G}^r_{g,d}$ which dominates $\mathcal{M}_g$), it is well known that if q is not 1 or 2, $k_{p,q} = 0$ except $k_{r-1,3} = h^1(L)$ and $k_{0,0} = 1$. Thus we only have to worry about the case $q = 1$ or 2. Also we have $k_{p,1} - k_{p-1,2}$ is a known constant only depending on g, r, d and p, thus it remains to determine $k_{p,1}$ or equivalently $k_{p-1,2}$ for $1 \leq p \leq r-1$.

Problem 1 seems to be too difficult to answer in its full generality at this point as one can see from some special case of it.

In the case $L = K_X$ the problem reduces to the generic Green conjecture, which was solved by Voisin and is essentially the only known case for all p besides some low genus examples.

Theorem 2. *(Voisin) Let X be a general curve of genus $g \geq 4$. Then $K_{i,2}(X, K_X) = 0$ for all $i \leq p$ if and only if $p < Cliff(X) = \lfloor \frac{g-1}{2} \rfloor$.*

For arbitrary g^r_d on a general curve X, even the simplest case to determine $k_{1,1}$, or equivalently $k_{0,2}$ is open. This is

Conjecture 3. *(Maximal rank conjecture for quadrics) For fixed g, r, d, let X be a general curve of genus g and $|L|$ be a general g^r_d on X, then the multiplication map*

$$S^2 H^0(X, L) \xrightarrow{\mu} H^0(X, L^2) \tag{1.1}$$

is of maximal rank i.e. either injective or surjective.

2 Main results

For conjecture 3, we manage to prove

Theorem 4. *(Wang) Let (X, L) be a general pair in $\mathcal{G}^r_{g,d}$ with $h^1(L) \leq 1$. Suppose*

$$d > \frac{5}{4} g + \frac{9}{4}, \quad \text{if } h^1(L) = 0, \text{ or}$$

$$d > \frac{5}{4} g + \frac{3}{4}, \quad \text{if } h^1(L) = 1,$$

then (X, L) is projectively normal (i.e $K_{0,q}(X, L) = 0$ for $q \geq 1$).

It is a very well known result of Green-Lazarsfeld that any very ample line bundle L on X with

$$deg(L) \geq 2g_X + 1 - 2h^1(L) - Cliff(X) \tag{2.2}$$

is projectively normal and the bound is sharp. Notice that (2.2) implies that $h^1(L) \leq 1$. If X is general,

$$Cliff(X) = \lfloor \frac{g_X - 1}{2} \rfloor,$$

thus Green-Lazarsfeld theorem predicts projective normality for general curves if d is bigger than roughly $\frac{3}{2}g$. Theorem 4 thus says that if we L is also general, we could improve the lower bound of d to roughly $\frac{5}{4}g$.

For higher syzygies, we could prove

Theorem 5. *(Wang) Let X be a general curves of genus g, L be a general g^r_d on X. Then*

1. *If $g \geq r+1$, $K_{p,1}(X, L) = 0$ for $p \geq \lfloor \frac{r+1}{2} \rfloor$.*
2. *If $h^1(L) = 1$ (which implies that $g \geq r+1$),*
   $K_{p-1,2}(X, L) = 0$ for $1 \leq p \leq r - \lfloor \frac{g}{2} \rfloor$, and
   $k_{p-1,2}(X, L) \leq (g - 2r + 2p - 1)\binom{r-1}{p-1}$ for $p > r - \lfloor \frac{g}{2} \rfloor$.

3 The idea of proof

Let's restrict ourselves to the proof of theorem 4. First notice that to prove projective normality for general (X, L), it suffices to show (1.1) is surjective (this is not trivial). Secondly we see that conjecture 3 depends on three discrete parameters g, r and h^1 (and therefore $d = g + r - h^1$). For each fixed pair of nonnegative integers r and h^1, we do induction on g. We start with $g = (r+1)h^1$ (i.e $\rho = 0$), and each step both g and d increase by 1. The key step in the inductive argument is

Theorem 6. *Let $X \subset \mathbb{P}^r$ be a general curve embedded by a general g^r_d $|L|$, and suppose one of the following two conditions holds*

1. *μ in (1.1) is injective, or*
2. *μ is surjective and there exists a quadric $Q \in \mathrm{Ker}(\mu)$ containing X but not containing the tangential variety $TX := \cup_{u\in X} T_u X$,*

then $(MRC)^r_{g+1,d+1}$ holds as well.

Theorem 4 then follows from theorem 6 and the projective normality of rational normal curves ($h^1 = 0$) and general canonical curves ($h^1 = 1$) provided that we can verify the hypothesis about TX in theorem 6 (2).

We use an infinitesimal methods to prove theorem 6. For a general g^r_d L' on a general curve C of genus g, suppose we have either $K_{p,1}(C, L') = 0$ or $K_{p-1,2}(C, L') = 0$. We construct a reducible nodal curve X_0 of genus $g+1$ by attaching on C an elliptic tail E and construct L_0 on X_0 of degree $d+1$ in a suitable manner such that $H^0(C, L') \cong H^0(X_0, L_0)$. There are two cases we need to take care of in the inductive argument. The easy case is when $K_{p,1}(C, L') = 0$, then $K_{p,1}(X_0, L_0) = 0$ by construction and then one could do induction on g by attaching elliptic tails one at a time. The more difficult case is that when $K_{p-1,2}(C, L')^\vee = K_{r-p,0}(C, L'; K_C) = 0$. In this case we do not have vanishing of $K_{r-p,0}(X_0, L_0; \omega_{X_0})$ but we can describe the generators of it explicitly and compute the obstructions for these classes to deform to nearby fibers (X_t, L_t). The upshot is that (X_0, L_0) are chosen carefully so that $K_{r-p,0}(X_0, L_0; \omega_{X_0})$ is generated by pure tensors in $\wedge^{r-p} H^0(L_0) \otimes H^0(\omega_{X_0})$ and therefore it is possible to control the obstructions. It turns out that the obstruction classes lies in $K_{r-p-1,1}(C, L'; K_C) = K_{p,1}(C, L')^\vee$.

When $p = 1$, under the assumption of theorem 6, we could fully control these obstructions and show that the extra syzygies in $K_{r-p,0}(X_0, L_0; \omega_{X_0})$ does not deform to nearby fiber and therefore $K_{r-p,0}(X_t, L_t; \omega_{X_t}) = 0$. This means that $(MRC)^r_{g,d}$ implies $(MRC)^r_{g+1,d+1}$ and therefore theorem 6 holds.