HARVEY MUDD COLLEGE

Gromov-Witten Theory of Toric Blowups

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Introduction

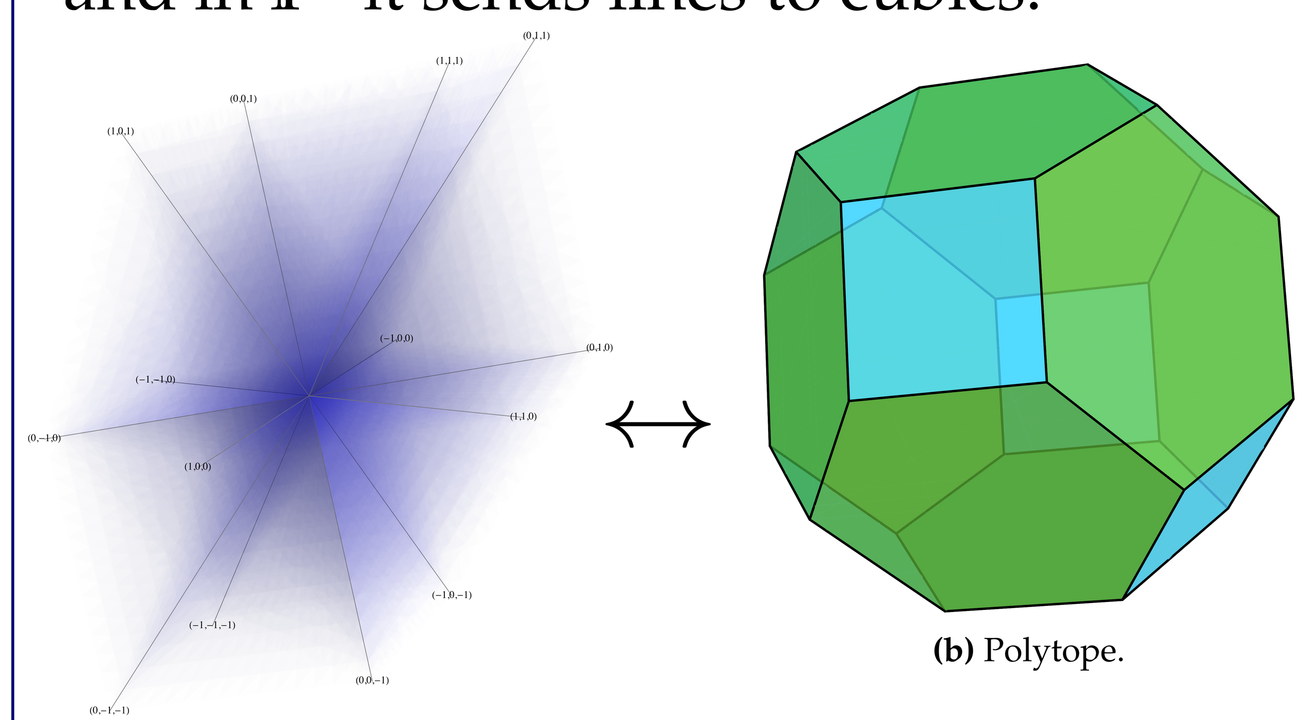
Toric varieties have been a shining example of success in Gromov-Witten theory. The Gromov-Witten theory of toric 3-folds can be computed by several techniques, including the topological vertex, (virtual) fixed point localization and the remodeling conjecture. Each of these techniques harnesses the combinatorics of the toric variety in a different way. Our motivation is to develop new computational techniques that stem from the combinatorics of these varieties.

Cremona Symmetry on $\mathbb{P}^n$

The classical Cremona transformation on $\mathbb{P}^n$ is the birational map given by coordinate-wise reciprocation,

$$(x_0 : \cdots : x_n) \mapsto (\frac{1}{x_0} : \cdots : \frac{1}{x_n})$$

As a map on the fan (or polytope) of $\mathbb{P}^n$, the Cremona symmetry is induced by reflection through the origin in the lattice $\mathbb{Z}^n$. The Cremona transformation is resolved by maximally blowing up the torus fixed subvarieties of $\mathbb{P}^n$. The polytope of this variety is known as the permutohedron Π_n (shown in figure 1 for $n = 3$). Göttsche-Pandharipande (in dimension 2) and Bryan-Karp (in dimension 3), showed that this map pushes forward to a nontrivial automorphism of the homology, and yields a nontrivial symmetry of the Gromov-Witten invariants. In $\mathbb{P}^2$ this map sends lines to conics, and in $\mathbb{P}^3$ it sends lines to cubics.



(a) Fan. $\leftrightarrow$ (b) Polytope.

Figure 1: The fan and polytope of the permutohedron.

A property of basic interest proved by Hu and Bryan-Karp, is that the symmetry of X_{Π_3}, descends to the blowup of $\mathbb{P}^3$ at six points. This gives the result enumerative significance. For instance, pushing forward the class of a line between two points in $\mathbb{P}^3$ via Cremona, we get

$$\langle\,\rangle^{\mathbb{P}^3(6)}_{0,h-e_5-e_6} = 1 = \langle\,\rangle^{\mathbb{P}^3(6)}_{0,3h-e_1-e_2-e_3-e_4-e_5-e_6}.$$

This recovers the result that there is exactly one cubic through 6 generic points in projective space.

Blowups of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Consider the cube (the polytope of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$). Blowing up points on an interior diagonal, and all lines intersecting these points, we are again left with the polytope Π_3. This common blowup for $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, gives us a birational map τ between these spaces, via blowup and blowdown.

Now, if σ is the map on the lattice polytope Π_3, given by the matrix

$$\sigma = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

then $\sigma_\star$ is a nontrivial symmetry of the homology on X_{Π_3} viewed as a blowup of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Descent of Toric Symmetry

The degeneration formula gives us the absolute invariants in terms of the relative ones,

$$\langle\,\rangle^X_{g,\beta} = \sum \sum_{\varphi_i \in H_\star(F)} \langle\,|\varphi_i\rangle^{(\hat{X}/F)}_{g',\hat{\beta}_1} \langle\,|\varphi^i\rangle^{(\hat{P}/F)}_{g'',\hat{\beta}_2},$$

where the sum is taken over curve splittings $\beta = \beta_1 + \beta_2$, and $\hat{P}$ is the projective completion of the normal bundle of the center of the blowup (in our case over a line). F is the relative divisor. Using relative invariants and deformation to the normal cone, we can prove that the symmetry $\sigma_\star$ above descends to a symmetry on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(2)$. There is an automorphism $\tau_\star$ on $A_\star(X_{\Pi_2})$, and thus, if we prove that the invariants on X_{Π_3} are equal to those on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1(2)$, together with the result of Bryan-Karp, we get a basic result relating the Gromov-Witten invariants on $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

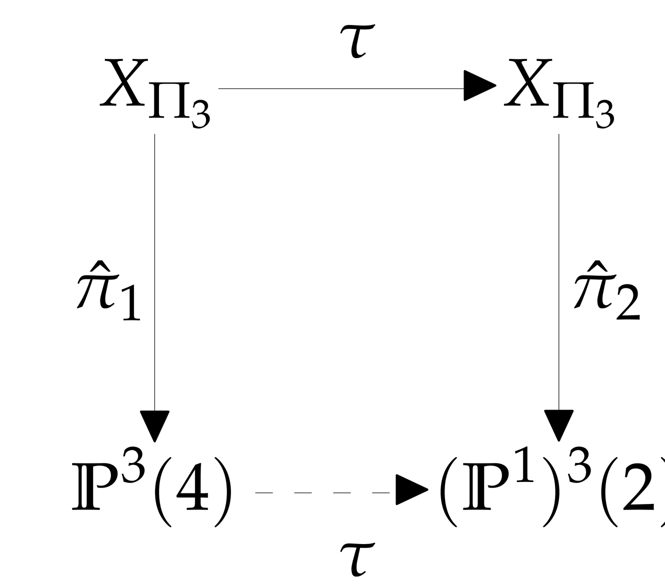


Figure 2: The birational map τ relates the GW invariants on $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Further there are the two involutions on X_{Π_3} that are nontrivial on classes pulled back from the base spaces. Let σ and τ are as before, and θ is the Cremona involution on $\mathbb{P}^3$. If β is a class on $\mathbb{P}^3(4)$ that lifts to a non exceptional class on X_{Π_3}, then we have

$$\langle\,\rangle^{\mathbb{P}^3(4)}_{g,\beta} = \langle\,\rangle^{\mathbb{P}^3(4)}_{g,\theta_\star\beta} = \langle\,\rangle^{(\mathbb{P}^1)^3(2)}_{g,\tau_\star\beta} = \langle\,\rangle^{(\mathbb{P}^1)^3(2)}_{g,\sigma_\star\tau_\star\beta}.$$

Note that GW invariants are not functorial under birational maps in general.

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