Nash Conjecture on Arcs for Surfaces

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ABSTRACT

In this poster we give the statement of Nash Conjecture on arcs for the surface case posed in [5] and sketch the recent proof in [3].

1 Surface singularities and resolution.

Let (X, O) be an isolated surface singularity embedded in some $(\mathbb{C}^K, O)$ for certain K defined by certain polynomial equations $f_1, ..., f_n \in \mathbb{C}[x_1, ..., x_k]$:

$$(X, O) := \{x \in (\mathbb{C}^k, O) : f_1(x) = ... = f_n(x) = 0\}.$$

A ***resolution of singularities*** of (X, O) is a smooth complex surface $\tilde{X}$ and a birrational proper morphism

$$\pi : \tilde{X} \to X$$

such that $\pi|_{\tilde{X} \setminus \pi^{-1}(O)}$ is an isomorphims onto $X \setminus \{O\}$.

- The space $\pi^{-1}(O)$ is a compact divisor E, that is a union of complex curves. Let $E_1, ..., E_r$ be the irreducible components.
- There exists a ***minimal resolution*** (any other resolution factors through it).

2 Arc spaces and its irreducible components.

A ***convergent arc*** is a germ of holomorphic mappings

$$\gamma : (\mathbb{C}, 0) \to (X, O).$$

It is given by k convergent power series with coefficients in $\mathbb{C}$,

$$(\sum_{j=1}^{\infty} a_{1,j}t^j, ..., \sum_{j=1}^{\infty} a_{k,j}t^j)$$

which is a zero of the equations of X in $\mathbb{C}^K$.

- The space of convergent arcs lies inside the inverse limit of n-jets. This limit is ***the space of arcs*** $\mathcal{X}_\infty$. It has an algebraic variety structure of infinite dimension and we consider its irreducible components.

Any arc $\gamma : (\mathbb{C}, 0) \to (X, O)$ admits a unique ***lifting*** $\tilde{\gamma}$ to $(\tilde{X}, O)$:

$$\begin{array}{ccc} & & (\tilde{X}, E) \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi \\ (\mathbb{C}, 0) & \xrightarrow{\gamma} & (X, O) \end{array}$$

To each divisor E_i one associates the following set of arcs:

$$N_i = \{\text{arcs whose lifting meets } E_i\}.$$

Its closure $\bar{N}_i$ is irreducible and $\mathcal{X}_\infty(X) = \bigcup_i \bar{N}_i$.

3 Statement of Nash Conjecture.

Consider the minimal resolution of (X, O).
Every set $\overline{N}_i$ is an irreducible component of $\mathcal{X}_\infty$.

In other words, no inclussion (or ***adjacency***) $N_i \subseteq \overline{N}_j$ is possible.

4 How can one characterize an adjacency $N_i \subseteq \overline{N}_j$ by means of arcs in $\mathcal{X}_\infty$?

- Curve Selection Lemma is not true in infinite dimensional spaces: points in the closure of a set might not be "approximated by curves in the set".

After works of M. Lejeune-Jalabert [4], A. Reguera [7] and J. Fernández de Bobadilla [2] we have the following:

$N_i \subseteq \overline{N}_j$ if and only if for every convergent arc $\gamma : (\mathbb{C}, 0) \to X$ in N_i with transversal lifting $\tilde{\gamma}$ through a smooth point of E_i, there exists a family of arcs α_s depending holomorphically on $s \in (\mathbb{C}, 0)$ such that $\alpha_0 = \gamma$, and $\tilde{\alpha}_s(0)$ is in E_j for every small enough $s \neq 0$.

- We call such a family of arcs ***a wedge realizing the adjacency*** and write $\alpha : (\mathbb{C}^2, O) \to (X, O)$. The arc α_0 is ***the special arc*** of the wedge and α_s for $s \neq 0$ ***the generic arcs***. The space of parameters of the family α_s can be assumed to be a disc D_δ in $\mathbb{C}$.

5 Sketch of the proof for the case of good minimal resolution (all divisors smooth with normal crossings).

Let $\pi : \tilde{X} \to X$ be the minimal resolution of singularities of X. Assume there is an adjacency $N_0 \subseteq \overline{N}_j$ corresponding to divisors E_0 and E_j. Then there exists a wedge α realizing the adjacency. Consider a representative $\alpha|_{\mathcal{U}}$ of α for $\mathcal{U}$ a neighbourhood of $(0, 0)$ in $\mathbb{C}^2$.

- We can assume that for any $s \in D_\delta$, the space $\mathcal{U}_s := \mathcal{U} \cap (\mathbb{C} \times \{s\})$ is diffeomorphic to a disc.

Consider the mapping

$$(\alpha|_{\mathcal{U}}, Id_{D_\delta}) : \mathcal{U} \to X \times D_\delta.$$

Let H be the surface image of this map.

- The arc representative $\alpha_0|_{\mathcal{U}_0}$ may be chosen such that the only preimage of the singular point is 0 but the arc representatives of the generic arc $\alpha_s|_{\mathcal{U}_s}$ may send several points to the singular point of X: ***the returns*** $\alpha_s|_{\mathcal{U}_s}^{-1}(O) = \{0, t_1, ..., t_q\}$ (appeared first in [6]).

  The image of the lifting $\tilde{\alpha}_s(\mathcal{U}_s)$ meets the exceptional divisor E in $q + 1$ points.

We consider the mapping

$$\sigma := (\pi, Id_{D_\delta}) : \tilde{X} \times D_\delta \to X \times D_\delta.$$

The pullback $\sigma^{-1}(H)$ is a divisor in the smooth 3-fold $\tilde{X} \times D_\delta$:

$$\sigma^{-1}(H) = Y + \sum_{i=0}^{r} n_i(E_i \times D_\delta),$$

where Y denotes the strict transform of H. The surface Y can be seen as a family of divisors in X over D_δ:

- For $s \neq 0$ the fibre Y_s is reduced and coincides with the image of the mapping $\tilde{\alpha}_s$. It touches E in E_j and possibly in other points corresponding to the returns.
- For $s = 0$ the divisor $Y_0 \subset X$ decomposes as

$$Y_0 = \tilde{\alpha_0}(\mathcal{U}_0) + \sum_{i=0}^{r} a_i E_i.$$

  The divisor Y_0 only touches E transversally in E_0.

5.2 Key point.

- The divisor Y_s is a reduced deformation of Y_0. Using this we can find an uppperbound for the euler characteristique of the normalization of Y_s that we denote again by $\mathcal{U}_s$.

  The upperbound will be in terms of:
  - the topology of Y_0,
  - the multiplicities a_i and
  - the set of intersection points of $Y_s \cap E$.

  In particular we will see that

$$\chi(\mathcal{U}_s) \leq 0. \qquad (1)$$

- The lifting $\tilde{\alpha}_s : \mathcal{U}_s \to Y_s$ is in fact the normalization of Y_s and $\mathcal{U}_s$ is a disc in $\mathbb{C}$. In particular $\chi(\mathcal{U}_s) = 1$ and contradiction!!.

5.3 A key example to find the upperbound (1).

Consider any reduced deformation F_s of the divisor F_0 defined by $x^a y^b = 0$ inside a Milnor ball in $\mathbb{C}^2$.

- The normalization of such a deformation is a disjoint union of riemann surfaces, all of them with boundary.
- Its euler characteristique is (upper) bounded by the number of discs.
- A first approximation: it has at most $a + b$ discs.
- If there is a disc in the normalization then its image in the ball will meet at least once the divisor $x^a y^b = 0$.
- The number of discs is then (upper) bounded by the number of intersection points of the deformation F_s and the divisor F_0.

*In our case these intersection points $F_s \cap F_0$ will correspond mainly to the returns of the generic arc of the wedge, that is to $Y_s \cap E$.

5.4 Euler characteristic upperbounds.

We finally get the following

$$\chi(\mathcal{U}_s) \leq a_0 - 1 + \sum_{i=0}^{r} a_i(\chi(\dot{E}_i)) + Y_s \cdot E. \qquad (2)$$

where $\dot{E}_i$ is the riemman surface E_i minus small balls centred at the intersection points with other divisors and with $\tilde{\alpha}_0(\mathcal{U}_0)$.

- By the invariance of the intersection number by deformations we have that

$$Y_s \cdot E = Y_0 \cdot E.$$

  It can be expressed in terms of the intersection matrix

$$M = (k_{ij})_{ij} = (E_i \cdot E_j)_{ij}.$$

- Observe that

$$\chi(\dot{E}_i) = 2 - 2g_i - \sum_{j \neq i} k_{ij} \text{ for any } i \neq 0 \text{ and}$$

$$\chi(\dot{E}_0) = 2 - 2g_0 - 1 - \sum_{j \neq 0} k_{0j}.$$

With these observations in (2) we get that

$$\chi(\mathcal{U}_s) \leq \sum_{i=0}^{r} a_i(2 - 2g_i + k_{i,i}). \qquad (3)$$

5.5 Final argument: the use of the minimality of the resolution.

We finally see that the upperbound in (3) is less or equal than 0. Recall the Castelnuovo Contractibility Criterion:

A smooth rational curve with self-intersection -1 in a smooth surface can be contracted.

- In particular such a curve can not be an irreducible component of the excepcional divisor of a minimal resolution.

We know that $k_{ii} \leq -1$. If $k_{ii} \leq -2$ then the i-summand in (3) is less or equal than 0. If $k_{ii} = -1$ then by the Castelnuovo criterion, since E_i is smooth by hypothesis, E_i can not be rational and we have $g_i > 0$. So every summand in (3) is less or equal than 0 and we finish the proof.

6 Sketch of the proof in the general case.

- In the general case, the essential divisors of the minimal resolutions may have singularities and the intersections between them may not be normal crossings.

Analogously to the good minimal resolution case we get the following inequality:

$$\chi(\mathcal{U}_s) \leq \sum_{i=0}^{r} a_i(2 - 2g_i - \mu_{E_i} - \eta_{E_i} + k_{i,i}) \qquad (4)$$

where, if $\{(\Gamma_k, p_k)\}_{k=1}^{d}$ is the set of local branches of E_i at the singular points of $Sing((Y_0)^{red}$, then we have that

μ_{E_i} is the sum of Milnor numbers of these local branches of E_i,

ν_{E_i} the number of these branches and

$\eta_{E_i} := \sum_{k=1}^{d} \sum_{l \neq k} I_{p_k}(\Gamma_k, \Gamma_l)$.

A similar argument using Castelnuovo Contraction Theorem allows us to conclude that $\chi(\mathcal{U}_s) \leq 0$ and we finish the proof.

7 Higher dimensional case.

For higher dimensional varieties there do not exist a minimal resolution but we can consider the essential components which appear in any resolution up to birational morphisms.
There are examples of varieties of dimension greater than 3 by J. Kollar and S. Ishii for which there are essential components that do not give irreducible components of the space of arcs.
The case of dimension three remains open.

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