Lattice-polarized K3 fibrations of Calabi-Yau threefolds

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Motivation

- Our work is motivated by Doran-Morgan classification [8] of integral variations of Hodge structure which can underlie one-parameter families of Calabi-Yau threefolds (with $h^{2,1} = 1$). They have found 14 real / 23 integral variations that can underlie such families and have provided geometric examples realizing 13 real / 21 integral classes.
- Trying to obtain examples for "missing" classes by mimicking other ones fails, the constructed Calabi-Yau threefolds (realized as anticanonical hypersurfaces or nef complete intersections in toric varieties) have $h^{2,1} = 3$, but certain subfamilies formally give the desired GKZ series.
- We had two models proposed for "the 14th case" and were able to match them using fibrations by M-polarized K3 surfaces. The behaviour of these fibrations led us to considering K3 fibrations of "the 23rd case" as well as of "regular" examples.

M-polarized K3 surfaces

Definition. *Let $M = H \oplus E_8 \oplus E_8$, where H is the hyperbolic lattice. An M-polarization on a K3 surface X is a primitive lattice embedding $i : M \hookrightarrow \mathrm{NS}(X)$, such that the image $i(M)$ in the Néron-Severi lattice $\mathrm{NS}(X)$ contains a pseudo-ample class (corresponding to an effective nef divisor with positive self-intersection).*

Theorem ([4, 7])**.** *If X is an M-polarized K3 surface, then*

1) *X is isomorphic to the minimal resolution of a quartic surface in $\mathbb{P}^3$ given by*
$$y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4) = 0;$$
2) *parameters a, b, and d in the above equation specify a unique point $(a, b, d) \in \mathbb{WP}(2, 3, 6)$ with $d \neq 0$;*
3) *X can be realized as an anticanonical surface in (a resolution of) the toric Fano threefold polar to $\mathbb{WP}(1, 1, 4, 6)$;*
4) *X canonically corresponds to a pair of elliptic curves;*
5) *modular parameters of X and $\{E_1, E_2\}$ are related by*
$\pi = j(E_1)j(E_2) = \frac{a^3}{d}, \quad \sigma = j(E_1) + j(E_2) = \frac{a^3 - b^2 + d}{d};$
6) *there are exactly two isomorphism classes of elliptic fibrations with sections on X, both can be torically induced.*

The 14th case

- To get an example for "the missing real class" one may try to start with a complete intersection in $\mathbb{WP}(1, 1, 1, 1, 4, 6)$, but it is not Fano and necessary mirror construction is unclear. Kreuzer and Scheidegger proposed a replacement: it leads to complete intersections Y_2 with $h^{2,1}(Y_2) = 3$ and $h^{2,1}_{\mathrm{poly}}(Y_2) = 2$.
- Alternatively, one may try to start with a hypersurface in a fourfold fibered by $\mathbb{WP}(1, 1, 4, 6)$, preferably "torically balanced", which is the case for $\mathbb{WP}(1, 1, 2, 8, 12)$. This leads to hypersurfaces Z_3 with $h^{2,1}(Z_3) = 3 = h^{2,1}_{\mathrm{poly}}(Z_3)$.
- Both models admit M-polarized K3 fibrations and have one-parameter subfamilies Y_1 and Z_1 that formally give the desired GKZ series. Using modular parameters of K3 fibers Y_2 can be "matched" with a two-parameter subfamily Z_2, while Z_3 can be easily compared to the work of Billo et. al. [1].
- We have determined singularities of (generic members of) subfamilies using Sage [9] and Magma [3]:
  - Y_1 has two nodes;
  - Z_2 has an elliptic curve C of cA_1 singular points;
  - on Z_1 the curve of singularities develops a node.
- It is obvious from the presence of singularities that Y_2 is not isomorphic to Z_2. However, extensively using toric tools in Sage [2], we have found a contraction morphism between these threefolds, which means that full (smooth) families Y_2 and Z_3 are connected via a geometric transition (of type III).
- The nodes of Y_1 cannot be crepantly resolved, so there is no geometric transition to Y_2 through Y_1.
- Modular parameters of fibers of Y_2 are
$$\pi = \frac{1}{12^6\alpha} \cdot \frac{t}{(t+1)^2},$$
$$\sigma = 1 + \frac{\beta - 3 \cdot 12^2\beta^2}{12^3\alpha} \cdot \frac{t}{(t+1)^2},$$
where t is an affine coordinate on the base and α and β are deformation parameters.
- The subfamily Y_1 is determined by the condition $\beta = 0$. Note that in this case $\sigma = 1$ and the image curve of fibers in the moduli space of M-polarized K3 surfaces is fixed, only its parametrization depends on the remaining parameter α.

The 23rd case

- An example for "the missing integral class" should come from the mirror octic twin: a hypersurface inside the quotient by a finite group of $\mathbb{WP}(1, 1, 1, 1, 4)$, obtained by considering its reflexive polytope in a more coarse lattice (but not as coarse as for the polar of this space). The corresponding anticanonical hypersurfaces Z_3 have $h^{2,1}(Z_3) = 3 = h^{2,1}_{\mathrm{poly}}(Z_3)$.
- One-parameter subfamily Z_1 formally yields the desired GKZ series, its generic members are smooth.
- There is a torically induced fibration by S-polarized K3 surfaces, where $S = H \oplus E_7 \oplus E_7$ is a sublattice of M and $N = H \oplus E_8 \oplus E_7$. Many properties of K3 surfaces with N and S polarization are similar to those of M-polarized ones [5, 6]. In particular:
  - they have the same normal forms with one extra monomial cxz^2w for N and two more beyond that for S: $exw^3 - \frac{f}{2}w^4$;
  - the monomial coefficients specify a unique point $(K_4, K_6, K_8, K_{10}, K_{12}) \in \mathbb{WP}(2, 3, 4, 5, 6)$ by
  $K_4 = a,\ K_6 = b,\ K_8 = ce,\ K_{10} = cf + de,\ K_{12} = df.$
- Fiber parameters for Z_3 are
$$K_4 = 3\alpha s, \quad K_6 = \beta s - \tfrac{1}{1728}, \quad K_8 = 36\alpha s^2,$$
$$K_{10} = -12\alpha s^2, \quad K_{12} = \alpha s^2, \quad s = t + \gamma + t^{-1},$$
where t is an affine coordinate on the base and α, β, γ are deformation parameters.
- The subfamily Z_1 is determined by the conditions $\beta = \gamma = 0$, leading to
$$s = t + t^{-1} \quad \text{and} \quad K_6 = -\tfrac{1}{1728}.$$
As we can see, the choice of the remaining parameter α changes the image curve of fibers.

Hypersurfaces in $\mathbb{WP}$'s.

- Four of the "known examples" of one-parameter families of Calabi-Yau threefolds are realized as hypersurfaces in polars of weighted projective spaces, all admitting M-polarized K3-fibrations. These polarizations may be enhanced to $M_n = M \oplus \langle -2n \rangle$.
- Due to the polarization enhancement, the image curve of fibers in the moduli space of M-polarized K3 surfaces must be the same throughout the family.
- For anticanonical hypersurfaces in the polar of $\mathbb{P}^4$ (fibered by the polar of $\mathbb{P}^3$) parameters of the M_2-polarized K3 fibers are given by
$$a = s + 1, \quad b = \frac{9}{2}s - 1, \quad d = s^3, \quad s = \alpha\frac{t^5}{(t+1)^4},$$
where t is an affine coordinate on the base and α is a deformation parameter.
- For $\mathbb{WP}(1, 1, 1, 2, 5)^\circ$ (fibered by $\mathbb{WP}(1, 1, 1, 3)^\circ$) we similarly get parameters of M_1-polarized K3 fibers as
$$a = 1, \quad b = s + 1, \quad d = s^2, \quad s = \alpha\frac{t^5}{(t-4)^3},$$
where t is an affine coordinate on the base and α is a deformation parameter.

References

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