QUARTIC SURFACES AS LINEAR PFAFFIANS

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Question 1

Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 2$. Does there exist a $d \times d$ matrix M of linear forms such that

$$X = \{\det M = 0\}?$$

- $d = 2$ or $d = 3$: **YES.** (The matrix M is induced by a ruling line on X for $d = 2$ and by a twisted cubic on X for $d = 3$.)

- $d \geq 4$: **ALMOST NEVER.** While the determinant of a sufficiently general $d \times d$ matrix M of linear forms in 4 variable cuts out a smooth surface $X \subseteq \mathbb{P}^3$ of degree d, the degeneracy locus of a $(d-1) \times d$ submatrix of M is a curve on X that is not a hypersurface section.

  But the Noether-Lefschetz Theorem implies that a very general hypersurface of degree d cannot admit such a curve.

In light of this, we consider a different question for surfaces of degree 4 or greater. Recall that if M is a $2d \times 2d$ skew-symmetric matrix, the **Pfaffian** of M is $\mathrm{Pf}(M) := \sqrt{\det M}$.

Question 2

Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 4$. Does there exist a $2d \times 2d$ skew-symmetric matrix M of linear forms such that

$$X = \{\mathrm{Pf}(M) = 0\}?$$

(Beauville-Schreyer,'00) The answer is **YES** if X is a **general** surface of degree $d \leq 15$ and **NO** for $d \geq 16$. The proof relies on a Macaulay 2 calculation which shows that the Pfaffian map from the space of $2d \times 2d$ skew-symmetric matrices of linear forms to $|\mathcal{O}_{\mathbb{P}^3}(d)|$ is dominant for $4 \leq d \leq 15$.

Theorem 1 (CKM)

For every smooth quartic surface $X \subseteq \mathbb{P}^3$, there exists an 8×8 skew-symmetric matrix M of linear forms such that

$$X = \{\mathrm{Pf}(M) = 0\}.$$

It is important for our proof that every smooth quartic X is a **K3 surface,** i.e. satisfies

$$\omega_X \cong \mathcal{O}_X, \quad H^1(\mathcal{O}_X) = 0$$

The strategy is to construct a rank-2 vector bundle $\mathcal{E}$ on X which must be the cokernel of an 8×8 skew-symmetric matrix of linear forms.

Proposition (Beauville '00)

Let $X \subseteq \mathbb{P}^3$ be a smooth surface of degree $d \geq 2$. Then the following are equivalent:

(i) There exists a $2d \times 2d$ skew-symmetric matrix M of linear forms such that $X = \{\mathrm{Pf}(M) = 0\}$.

(ii) There exists a rank-2 vector bundle $\mathcal{E}$ on X with $\wedge^2\mathcal{E} \cong \mathcal{O}_X(d-1)$ and $c_2(\mathcal{E}) = \frac{d(d-1)(2d-1)}{6}$ which is ACM, i.e. satisfies the vanishings

$$H^1(X, \mathcal{E}(m)) = 0 \quad \forall m \in \mathbb{Z}$$

One can try to produce such a bundle $\mathcal{E}$ by taking a smooth curve $C \in |\mathcal{O}_X(d-1)|$ and a globally generated line bundle $\mathcal{L}$ of degree $\frac{d(d-1)(2d-1)}{6}$ with $h^0(\mathcal{L}) = 2$ (if it exists!) and defining $\mathcal{E}$ by the exact sequence

$$0 \to \mathcal{E}^\vee \to H^0(\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L} \to 0$$

We are concerned with the case $d = 4$, so we want a smooth curve C in the linear system $|\mathcal{O}_X(3)|$ (which is a smooth complete intersection of type (3,4) in $\mathbb{P}^3$) and a globally generated line bundle $\mathcal{L}$ of degree 14 satisfying $h^0(\mathcal{L}) = 2$ such that the vector bundle $\mathcal{E}$ in the exact sequence

$$0 \to \mathcal{E}^\vee \to H^0(\mathcal{L}) \otimes \mathcal{O}_X \to \mathcal{L} \to 0$$

is ACM. **The following issues must be addressed:**

- While standard Brill-Noether theory guarantees plenty of line bundles $\mathcal{L}$ of degree 14 on C with $h^0(\mathcal{L}) = 2$, it does **not** guarantee that any of them are globally generated.
- Even if we can find a globally generated $\mathcal{L}$, the resulting vector bundle $\mathcal{E}$ might not be ACM.

Recall that for a smooth projective curve C and positive integers r, d,

$$W^r_d(C) := \{\mathcal{L} \in \mathrm{Pic}^d(C) : h^0(\mathcal{L}) \geq r+1\}$$

If C is a smooth complete intersection curve of type (3,4), the dimension of $W^1_{14}(C)$ is at least 7, and the general member $\mathcal{L}$ of $W^1_{14}(C)$ satisfies $h^0(\mathcal{L}) = 2$.

The locus in $W^1_{14}(C)$ parametrizing line bundles that are **not** globally generated is the image of the map

$$\sigma : C \times W^1_{13}(C) \to W^1_{14}(C), \quad (p, \mathcal{L}') \mapsto \mathcal{L}'(p)$$

Thus we can deduce the existence of a basepoint-free member of $W^1_{14}(C)$ by a dimension count if we can verify that $\dim W^1_{13}(C) \leq 5$.

We use a recent result on K3 surfaces to obtain this upper bound. If C is a smooth projective curve, then the *Clifford index* of C is defined to be

$$\mathrm{Cliff}(C) := \min\{d - 2r : \exists \text{ special } \mathcal{L} \ni c_1(\mathcal{L}) = d, h^0(\mathcal{L}) = r + 1 \geq 2\}$$

Clifford's Theorem implies that $\mathrm{Cliff}(C)$ is nonnegative, and is zero precisely when C is hyperelliptic.

Theorem (Aprodu-Farkas '11)

Let X be a K3 surface, and let L be a globally generated line bundle on X such that $\mathrm{Cliff}(C)$ is computed by a pencil of degree k for general smooth $C \in |L|$. Assume further that $k \leq \frac{L^2}{4} + \frac{3}{2}$. Then for general smooth $C \in |L|$,

$$\dim W^1_{n+k}(C) \leq n \text{ for } 0 \leq n \leq \frac{L^2}{2} + 3 - 2k.$$

We would like to apply this result to complete intersection curves of type (3,4). Fortunately, we have the following result on complete intersection curves in $\mathbb{P}^3$:

Theorem (Basili '96)

If $C \subseteq \mathbb{P}^3$ is a smooth complete intersection curve of type (m, n) for $(m, n) \neq (3, 3)$ and ℓ is the maximum number of collinear points on C, then $\mathrm{Cliff}(C)$ is computed by a pencil of degree $mn - \ell$.

Bézout's Theorem implies that if $X \subseteq \mathbb{P}^3$ is a smooth quartic and C is a general smooth member of $|\mathcal{O}_X(3)|$, then $\ell = 4$. So for all such C, Basili's theorem implies that $\mathrm{Cliff}(C)$ is computed by a pencil of degree 8, and we can apply the Aprodu-Farkas theorem to conclude that $\dim W^1_{13}(C) \leq 5$ as desired.

After a bit more work, we are able to conclude the following:

Proposition (CKM)

If $X \subseteq \mathbb{P}^3$ is a smooth quartic surface, there exists a 14-dimensional family $\mathcal{Y}$ of simple vector bundles of rank 2 on X such that each $\mathcal{E} \in \mathcal{Y}$ satisfies the following properties:

- $\wedge^2\mathcal{E} \cong \mathcal{O}_X(3)$ and $c_2(\mathcal{E}) = 14$.
- $H^1(\mathcal{E}(m)) = 0$ for all $m \leq -3$ and all $m \geq 0$.

Note that the members of $\mathcal{Y}$ are not necessarily ACM. However, Serre duality combined with the isomorphism $\mathcal{E} \cong \mathcal{E}^\vee(3)$ yields the following

Observation

A vector bundle $\mathcal{E} \in \mathcal{Y}$ is ACM if and only if $H^1(\mathcal{E}^\vee(-1)) = 0$.

Our final task is to show that the general member of $\mathcal{Y}$ is ACM. To clarify the obstruction to $\mathcal{E} \in \mathcal{Y}$ being ACM, we look to length-14 subschemes Z of X which represent $c_2(\mathcal{E})$. Since $\mathcal{E}$ might not be globally generated, the existence of such Z is not immediate.

Proposition (CKM)

For all $\mathcal{E} \in \mathcal{Y}$, there exists an l.c.i. subscheme Z of X and an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{I}_{Z|X}(3) \to 0$$

Moreover, if $\mathcal{E} \in \mathcal{Y}$ fits into such a sequence, then $\mathcal{E}$ is ACM if and only if Z does not lie on a quadric.

The proof of our theorem is concluded by using the fact that any Z in such a sequence satisfies the Cayley-Bacharach property with respect to $\mathcal{O}_X(3)$, and showing by way of a dimension count that the length-14 subschemes of X which are Cayley-Bacharach with respect to $\mathcal{O}_X(3)$ and lie on a quadric form a locus "too small" to come from the general member of $\mathcal{Y}$.

The fact that the members of $\mathcal{Y}$ are all **simple** vector bundles is important, as it guarantees that $\mathcal{Y}$ has the "correct" dimension. For many quartic surfaces, our bundles satisfy a stronger indecomposability property:

Proposition (CKM)

If $X \subseteq \mathbb{P}^3$ is a smooth quartic surface with Picard number not equal to 2, and X contains only finitely many smooth rational curves, then the general member $\mathcal{E} \in \mathcal{Y}$ is $\mu-$stable with respect to $\mathcal{O}_X(1)$.

It is likely that a different approach is required for surfaces of degree 5 through 15, since our method depends heavily on the K3 aspect of smooth quartic surfaces. However, it seems reasonable to ask

Question 3

Can our method be applied to produce other types of indecomposable ACM bundles on K3 surfaces, possibly of higher rank?

Smooth curves on K3 surfaces whose Clifford index is **not** computed by a pencil are relatively rare and have been completely classified by Knutsen '07, so it is quite possible that we can use the Aprodu-Farkas theorem to construct other ACM bundles.

THANKS FOR STOPPING BY!!!