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Group Actions on Hyperplane Complements

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Configuration Spaces & Equivariant Cohomology

Let $C_n(\mathbb{R}^k)$ be the configuration space of n distinct points in $\mathbb{R}^k$. The group S_n acts on $C_n(\mathbb{R}^k)$ by permuting the points. This action induces an action of S_n on $H^*(C_n(\mathbb{R}^k))$.

Proposition 1.1. As an S_n-representation, $H^*(C_n(\mathbb{R}^3))$ is isomorphic to the regular representation.

In fact, this proposition is true for $C_n(\mathbb{R}^k)$ where k is odd, but we will focus on the $k = 3$ case. There is a very nice proof of this result using equivariant cohomology.

Let $T = S^1$, let X be a T-space, and consider the Serre spectral sequence of the associated fiber bundle over BT with fiber X. We say that X is **equivariantly formal** if this spectral sequence collapses at the E_2 page $H^*(BT) \otimes H^*(X)$.

Proposition 1.2. [Bo, GKM] Let X be an equivariantly formal T-space, and let $F = X^T$. Then

$$H_T^*(X)/\langle u\rangle \cong H^*(X) \text{ and}$$
$$H_T^*(X)/\langle u-1\rangle \cong H^*(F).$$

Thus $H^*(F)$ admits a filtration with

$$\operatorname{gr} H^*(F) \cong H^*(X).$$

Remark 1.3. If a group W acts on X commuting with the T-action, then these isomorphisms are W-equivariant.

Let T act on $\mathbb{R}^3$ by rotation about the x-axis. This induces an action of T on $C_n(\mathbb{R}^3)$ with fixed point set $C_n(\mathbb{R})$. Note that this action of T commutes with the action of S_n.

One way to think of $C_n(\mathbb{R})$ is as the complement of the reflecting hyperplanes of the root system whose Weyl group is S_n. The cohomology of this space is generated in degree 0, and has a basis in bijection with the chambers. As an S_n-representation, this is clearly the regular representation. Finally, since the category of S_n representations is semisimple, $H^*(C_n(\mathbb{R}^3)) \cong \operatorname{gr} H^*(C_n(\mathbb{R}))$ is isomorphic to the regular representation as well.

Arrangement Complements

In this section, we will generalize Proposition 1.1 to the setting of real hyperplane arrangements.

Definition 2.1. Let V be a finite dimensional real vector space, and let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a hyperplane arrangement in V given by $H_i = \omega_i^{-1}(a_i)$ for some non-zero linear form $\omega_i : V \to \mathbb{R}$. Define linear maps $\omega_{i,k} : V^k \to \mathbb{R}^k$ by

$$\omega_{i,k}(v_1, \dots, v_k) = (\omega_i(v_1), \dots, \omega_i(v_k)).$$

The space $M_k(\mathcal{A})$ is defined to be the complement of the union of the affine subspaces

$$\omega_{i,k}^{-1}(a_1, 0, \dots, 0).$$

In this section, we will focus on $M_3(\mathcal{A})$, as $H^*(M_k(\mathcal{A})) \cong H^*(M_3(\mathcal{A}))$ for k odd and $k > 3$.

Example 2.2. If $\mathcal{A}$ is the braid arrangement in $\mathbb{R}^n$, then $M_3(\mathcal{A})$ is the configuration space $C_n(\mathbb{R}^3)$.

We may define an action of $T = S^1$ on the space $M_k(\mathcal{A})$ by extend the action on $\mathbb{R}^3$ to

$$V^3 \cong V \otimes (\mathbb{R} \oplus \mathbb{C}).$$

Note that $\omega_{i,3}$ is T-equivariant and $0 \in \mathbb{R}^3$ is T-fixed, so T acts on

$$M_3(\mathcal{A}) = \bigcap_{i=1}^{n} \omega_{i,3}^{-1}(\mathbb{R}^3 \setminus \{0\}).$$

The fixed point set of this action is $M_1(\mathcal{A})$ which is the complement of the real arrangement $\mathcal{A}$.

Proposition 2.3. [Mo] With respect to the filtration coming from equivariant cohomology

$$\operatorname{gr} H^0(M_1(\mathcal{A})) \cong H^*(M_3(\mathcal{A})).$$

The most interesting class of examples we have are Weyl groups acting on Coxeter arrangements.

Example 2.4. Let $\mathcal{A}$ be a Coxeter arrangement with Weyl group W. In this case W acts simply transitively on the chambers of $\mathcal{A}$, so $H^0(M_1(\mathcal{A}))$ is isomorphic to the regular representation. Since W is finite, its representation category is semisimple, so

$$H^*(M_3(\mathcal{A})) \cong \operatorname{gr} H^0(M_1(\mathcal{A}))$$

is isomorphic to the regular representation, as well.

Example 2.5. Consider the braid arrangement in $\mathbb{R}^4$, whose Weyl group is the symmetric group S_4. Computing each graded component as an S_4-representation we get

$$H^0(C_4(\mathbb{R}^3)) \cong \tau$$
$$H^2(C_4(\mathbb{R}^3)) \cong \rho + \Lambda^2(\rho)$$
$$H^4(C_4(\mathbb{R}^3)) \cong \rho + \Lambda^2(\rho) + 2\omega + \sigma$$
$$H^6(C_4(\mathbb{R}^3)) \cong \rho + \Lambda^2(\rho)$$

where τ corresponds to the partition (4), ρ corresponds to the partition $(3,1)$, $\Lambda^2(\rho)$ corresponds to the partition $(2,1,1)$, ω corresponds to the partition $(2,2)$, and σ corresponds to the partition $(1,1,1,1)$. Adding up the representations of all of the graded components yields the regular representation of S_4.

Question 2.6. For arbitrary n, what is the grading we obtain on the regular representation of S_n?

Filtration

The filtration on $H^0(M_1(\mathcal{A}))$ arising from Proposition 1.2 has been studied before. Let

$$H_i^+ = \{v \in V \mid \omega_i(v) > 0\},$$

and let

$$H_i^- = \{v \in V \mid \omega_i(v) < 0\}.$$

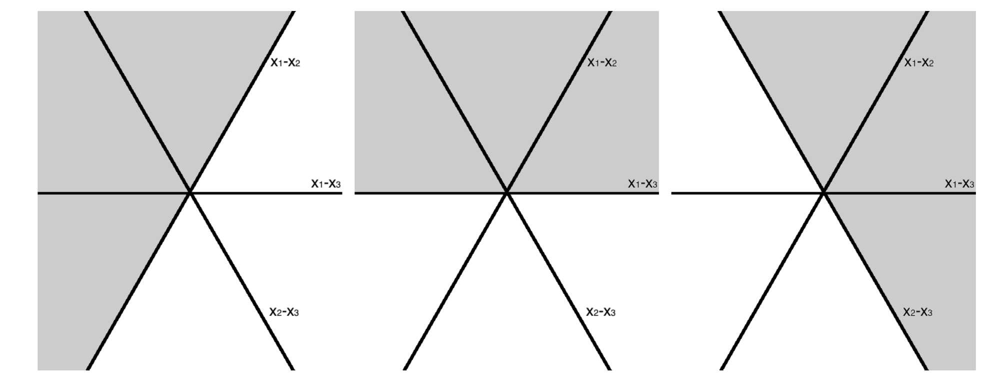
Define the Heaviside function $x_i \in H^0(M_1(\mathcal{A}))$ by putting

$$x_i(v) = \begin{cases} 1 & v \in H_i^+ \\ 0 & v \in H_i^-. \end{cases}$$

Then $H^0(M_1(\mathcal{A}))$ is generated by Heaviside functions; Varchenko and Gelfand consider the filtration by degree in these generators [VG].

Proposition 3.1. [Mo] The filtration on $H^0(M_1(\mathcal{A}))$ arising from Proposition 1.2 coincides with the Varchenko-Gelfand filtration.

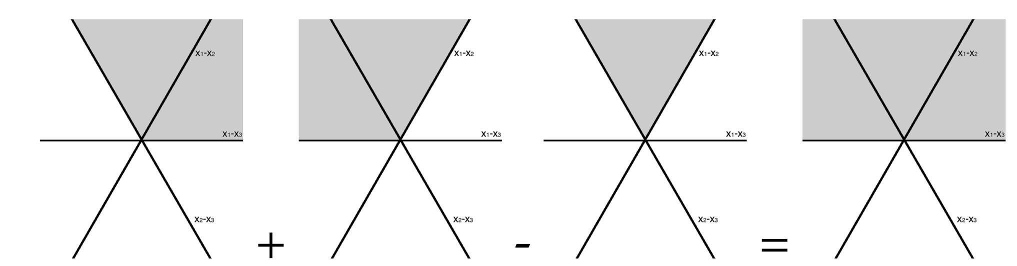
Example 3.2. Consider the example of the braid arrangement in $\mathbb{R}^3$. In our pictures, we will mod out by the diagonal copy of $\mathbb{R}$, which is contained in all of the hyperplanes. The ring $H^0(C_3(\mathbb{R}))$ is generated by the following Heaviside functions:



(a) x_{12} **(b)** x_{13} **(c)** x_{23}

Note that the x_{ij} are idempotent, and thus satisfy $x_{ij}^2 - x_{ij} = 0$. Also note that

$$x_{13}x_{23} + x_{12}x_{13} - x_{12}x_{23} = x_{13}.$$



These are the only relations between the Heaviside functions. By Proposition 1.2, the equivariant cohomology ring of $C_3(\mathbb{R}^3)$ is isomorphic to the Rees algebra of this filtered ring.

The equivariant cohomology ring of $C_3(\mathbb{R}^3)$ has presentation

$$\mathbb{Q}[x_{12}, x_{13}, x_{23}, u]/\mathcal{I}$$

where

$$\mathcal{I} = \langle x_{ij}(x_{ij} - u), x_{13}x_{23} + x_{12}x_{13} - x_{12}x_{23} - x_{13}u \rangle.$$

More generally, for any real arrangement $\mathcal{A}$, $H_T^*(M_3(\mathcal{A}))$ is isomorphic to the Rees algebra of the Varchenko-Gelfand filtration of $H^0(M_1(\mathcal{A}))$.

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