# Varieties Fibered by Good Minimal Models

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## Classification of Algebraic Varieties

For a given smooth projective variety $X$, there are many other varieties $X'$ birational to $X$, e.g. the blow-ups of $X$. It is then natural to consider the birational classification of varieties. For this purpose, it is desirable to find a "good" representative amongst the birational classes of $X$.

## Surfaces

Let $X$ be a smooth projective surface. In order to classify algebraic surfaces, we do the following:

- Suppose that $X$ contains a $(-1)$-curve $C$, i.e. $C \cong \mathbb{P}^1$ with $C^2 = -1$. Then by Castelnuovo's Theorem, one can contract this $(-1)$-curve by a birational morphism $X \to X'$ to a smooth surface $X'$. Since each time the Picard number drops by one, this process terminates and we end up with a smooth surface $X_{\min}$ with no $(-1)$-curves.
- If $\kappa(X) \geq 0$, then $K_{X_{\min}}$ nef. In fact, one can that $|mK_{X_{\min}}|$ is base point free for some $m > 0$.
- If $\kappa(X) = -\infty$, then $X_{\min}$ is either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle, In particular, $X_{\min}$ (and hence $X$) is covered by $\mathbb{P}^1$'s.

This is known by Italian school and has led to the Eriques classification of algebraic surfaces.

## Minimal Model Program

The minimal model program aims to generalize the procedure in the case of surfaces to higher dimensions. A similar picture is also expected to be true:

### Minimal model Conjecture

For any given smooth projective variety $X$, there exists a birational map $\phi : X \dashrightarrow X'$ such that

- The map $\phi$ extracts no divisors and is $K_X$-negative.
- If $\kappa(X) \geq 0$, then $|mK_{X'}|$ is base point free for some $m > 0$.
- If $\kappa(X) = -\infty$, then $X'$ has a structure of Mori fiber space. In particular, $X$ is covered by $\mathbb{P}^1$'s.

**We say that there exists a good minimal model for a variety $X$ if the minimal model conjecture holds for $X$.**

The minimal model conjecture is established in several cases:

- For threefolds by S. Mori, and others;
- For varieties of general type by Birkar-Cascini-Hacon-M$^c$Kernan.

For varieties of dimension higher than three of non-maximal Kodaira dimensions, i.e. $\kappa(X) < \dim X$, the conjecture is still open. In this work we focus on varieties of non-negative Kodaira dimension, i.e. $\kappa(X) \geq 0$.

## Reduction Theorem I

For a variety with $\kappa(X) \geq 0$, the Iitaka fibration is a map defined by $|mK_X|$ for $m$ sufficient divisible. The general fiber of the Iitaka fibration has Kodaira dimension zero.

### Theorem A

Let $X$ be a smooth projective variety of intermediate Kodaira dimension, i.e. $0 < \kappa(X) < \dim X$. Suppose that the general fiber $F$ of the Iitaka fibration has a good minimal model, then so does $X$.

As a corollary, we get immediately the following:

**The minimal model conjecture is reduced to the existence of good minimal models for varieties of Kodaira dimension zero and the non-vanishing conjecture.**

## Reduction Theorem II

If $\kappa(X) = 0$, then by a theorem of Kawamata the Albanese morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ is a surjective morphism with connected fiber. Moreover, if we assume the general fiber has a good minimal model, then by Kawamata's result on the Iitaka's conjecture C the general fiber of $\mathrm{alb}_X$ has Kodaira dimension zero. In this set up, we can do more:

### Theorem B

Let $X$ be a smooth projective variety of Kodaira dimension zero. If the general fiber $F$ of the Albanese morphism has a good minimal model, then so does $X$.

**In particular, the minimal model conjecture is reduced to the existence of good minimal models for regular varieties of Kodaira dimension zero, i.e. varieties $X$ with $\kappa(X) = h^1(X, \mathcal{O}_X) = 0$, and the nonvanishing conjecture.**

## Algebraic fiber spaces

An **algebraic fiber space** is a surjective morphism $f : X \to Y$ of normal varieties with connected fibers. Typical examples are the Iitaka fibrations or the Stein factorization of a proper morphism. In the spirit of the theorems proved above, we ask the following question:

### Question

Let $f : X \to Y$ be an algebraic fiber space. Assume the existence of good minimal model for the general fiber $F$ and the base $Y$, then does $X$ have a good minimal model?

Along this direction, we show that:

### Theorem C

Let $X$ be a smooth projective variety. Assume that the general fiber $F$ of the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ has $\dim F \leq 3$ (and hence $F$ has a good minimal model), then $X$ has a good minimal model.

## Technical Lemma

In general, the difficulty for producing a minimal model by running the minimal model program is that we don't know if the program terminates or not. The hypothesis on the existence of good minimal model enables us to prove the termination of a special minimal model program which is the key to prove the above mentioned theorems.

### Termination Lemma

Suppose that $X$ has a good minimal model, then any minimal model program with scaling of an ample divisor terminates.

## A Special Non-vanishing Theorem

One of the most important missing puzzle in the minimal model conjecture is the following:

### Non-vanishing Conjecture

A smooth projective variety $X$ with $K_X$ being pseudo-effective has non-negative Kodaira dimension.

We show the following non-vanishing result along this work:

### Special Non-vanishing Theorem

Let $X$ be a smooth variety and $F$ be the general fiber of the Stein factorization of the Albanese morphism. If $\kappa(F) \geq 0$, then $\kappa(X) \geq 0$.

This is achieve by the theory of Fourier-Mukai transforms and the generic vanishing theorem.

## Reference

[BCHM] C. Birkar, P. Cascini, C. Hacon, and J. McKernan: *Existence of minimal models for varieties of log general type.* J. Amer. Math. Soc. 23 (2010), No. 2, 405âĂŞ 468.

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