When is a variety a quotient of a smooth variety by a finite group?

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Background

Any variety of the form $X = U/G$, with U a smooth variety and G a finite group, must have quotient singularities. William Fulton posed the question, "**Is every variety with quotient singularities a global quotient of a smooth variety by a finite group?**" Removing the finiteness hypothesis, the answer is known to be "yes."

Theorem (Eddidin-Hassett-Kresch-Vistoli)

If X is a separated variety with quotient singularities over a field of characteristic 0, then $X = U/G$, where U is a smooth variety and G is a linear algebraic group.

Removing the "group hypothesis," the answer is also known to be "yes."

Theorem (Kresch-Vistoli + above result)

If X is an irreducible quasi-projective variety with quotient singularities over a field of characteristic 0, then there is a finite flat surjection from a smooth variety $U \to X$.

Main Result

Note that even for toric varieties, the answer to Fulton's question is not clear since the smooth variety may not be toric. For example, there does not exist a smooth toric variety U with an action of a finite group G so that $U/G \cong Bl(\mathbb{P}(1,1,2))$.

Though we do not yet have a complete answer to Fulton's question, we do have the following positive result.

Theorem

Suppose X is a quasi-projective variety with (tame) quotient singularities over an infinite field. If X is a quotient of a smooth variety by $\mathbb{G}_m^r$ (acting with finite stabilizers), then it is a quotient of a smooth variety by a finite group.

In particular, every quasi-projective (tame) simplicial toric variety is a global quotient of a smooth variety by a finite group.

The ingredients of the proof are the following well-known facts.

(1) A morphism of algebraic stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is representable if and only if it induces inclusions of stabilizers at geometric points.

(2) A quotient stack structure $\mathcal{X} \cong [U/G]$, with U an algebraic space, is equivalent to the data of a representable morphism $\mathcal{X} \to BG$.

(3) A morphism to BGL_r (resp. $B\mathbb{G}_m$, resp. $B\mu_n$) is equivalent to the data of a rank r vector bundle (resp. line bundle, resp. n-torsion line bundle), and the action of the stabilizer at a geometric point is given by the induced morphism to GL_r (resp. $\mathbb{G}_m$, resp. μ_n).

Key Observation

Combining (1), (2), and (3), we see that a smooth stack $\mathcal{X}$ is a quotient of a smooth algebraic space U by GL_r (resp. $\mathbb{G}_m^r$, resp. $\prod \mu_{n_i}$) if and only if it has a vector bundle $\mathcal{E}$ (resp. a sum of line bundles $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{L}_i$, resp. a sum of torsion line bundles $\mathcal{E} = \bigoplus \mathcal{L}_i$) so that the stabilizers at geometric points act faithfully on fibers.

Sketch Proof of Theorem

Suppose $X = V/\mathbb{G}_m^r$. Let $\mathcal{X} = [V/\mathbb{G}_m^r]$. Then we have line bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$ so that the stabilizers act faithfully on the fibers of $\oplus \mathcal{L}_i$. Cleverly choose an integer n and sections s_{ij} of $\mathcal{L}_i^{\otimes n}$. Let $\mathcal{Y}$ be the *n-th root stack* of $\mathcal{X}$ along the sections s_{ij}. This stack, by its universal property, comes equipped with line bundles $\mathcal{M}_{ij}$ so that $\mathcal{M}_{ij}^{\otimes n} \cong \mathcal{L}_i^{\otimes n}$ for each i. Moreover, the coarse space of $\mathcal{Y}$ is the same as the coarse space of $\mathcal{X}$, namely X. Now we have that $\mathcal{M}_{ij} \otimes \mathcal{L}_i^*$ are torsion line bundles on $\mathcal{Y}$. Because of your clever choice of n and s_{ij}, the stabilizers of $\mathcal{Y}$ act faithfully on the fibers of $\bigoplus_{i,j} \mathcal{M}_{ij} \otimes \mathcal{L}_i^*$. It follows that $\mathcal{Y} \cong [U/G]$, where U is a smooth variety and $G = \prod \mu_n$. Thus, $X = U/G$. □

Simplicial Toric Varieties

It turns out that an n-th root stack along an n-th power $\mathcal{L}^{\otimes n}$ has a nice description: it is the quotient of $\{t^n = s\} \subseteq \mathbb{V}(\mathcal{L})$ by the natural action of μ_n. This observation allows us to very explicitly generate the U and G in the theorem. Specifically, for simplicial toric varieties, we have a following procedure.

(0) Let X be a quasi-projective simplicial toric variety with singular locus Z. Let $X = V/G$ be its Cox construction.

(1) Choose Weil divisors $D_1, \dots, D_r$ which generate the class groups of all torus-invariant open affine subvarieties of X. Let n be an integer so that nD_i is Cartier for each i. The D_i can be chosen so that nD_i is very ample. Let $W \to X$ be the "total space of $\oplus D_i$," a toric variety which has irreducible torus-invariant divisors D_{e_i} linearly equivalent to the pullbacks of the D_i.

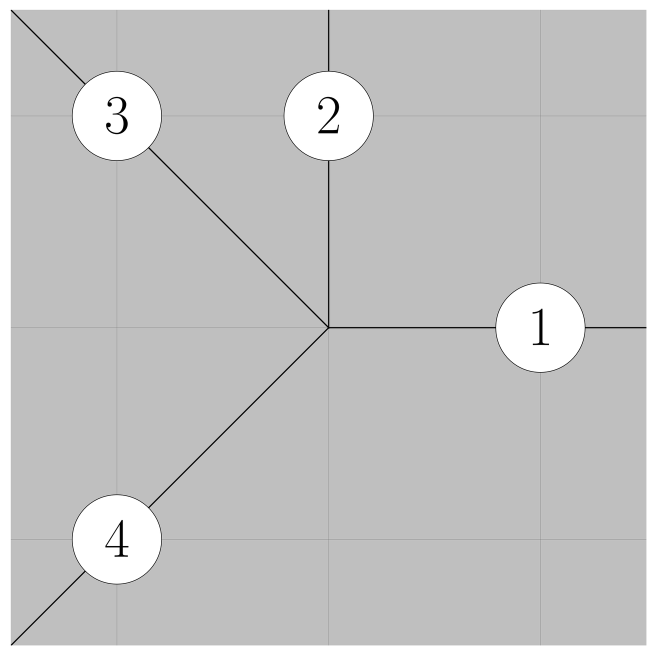
(2) Choose sections $\{s_{i,j}\}_{1\le j\le c_i}$ of $\mathcal{O}_X(nD_i)$ whose pullbacks to V have vanishing loci which are smooth, have simple normal crossings, and so that for each i, $\cap_j \{s_{i,j} = 0\}$ is disjoint from Z.

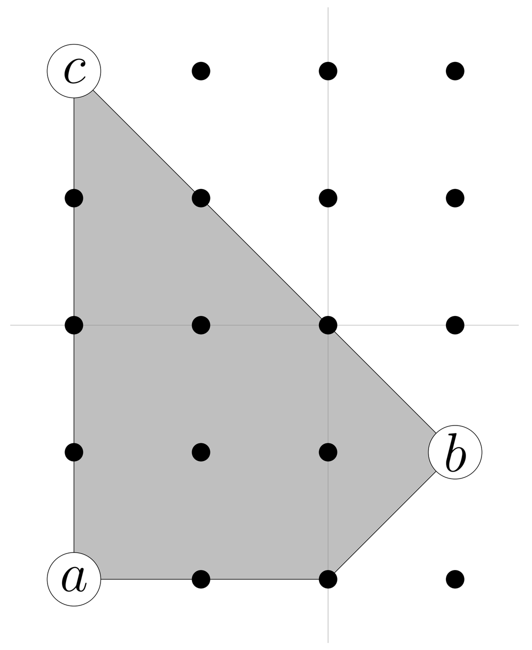
(3) Let $\psi_i\colon W \to \mathbb{P}^N$ be the map to projective space given by the divisor nD_{e_i}. Let s_i be a section of $\mathcal{O}_W(nD_{e_i})$ inducing the divisor nD_{e_i}, and let $\hat{s}_{i,j}$ be the pullbacks of the $s_{i,j}$ to W. Let Y_i be the intersection of the hyperplane sections $\{s_i - \hat{s}_{i,j}\}$, together with the action of $\mu_n^{c_i}$.

(4) The intersection $Y = \cap_i Y_i$ in W is a smooth variety such that $X \cong Y/\prod_i \mu_n^{c_i}$.

Example: $Bl(\mathbb{P}(1,1,2))$

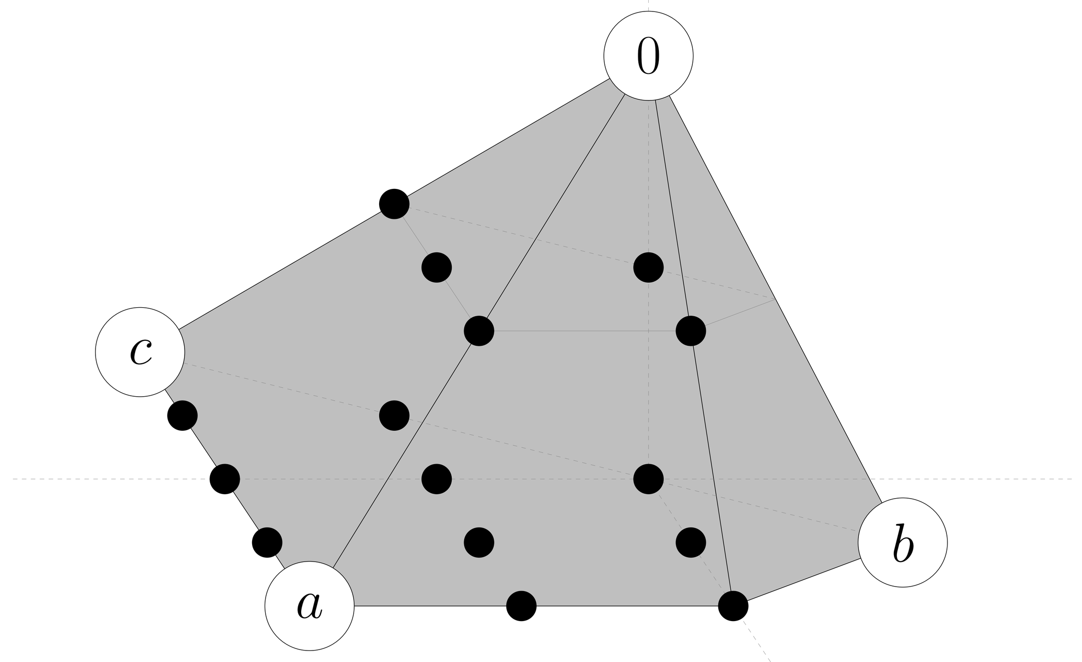
Let X be the toric variety with the following fan.





The divisor $D = D_1 + D_2 + D_3$ generates the class group of each torus-invariant open affine. We have that $2\cdot D$ is a very ample Cartier divisor, whose polytope is shown on the right. The lattice points in the polytope are torus semi-invariant global sections of $\mathcal{O}(2D)$. The sections s_a, s_b, and s_c corresponding to the labeled lattice points pull back to the monomials $x_3^2x_4^4$, $x_1^3x_2$, and $x_2^4x_3^6$ on $\mathbb{A}^4$, respectively. We check that the vanishing locus of $s = s_a + s_b + s_c$ is smooth and misses the singular locus.

The variety W is given by the fan obtained by adding a ray in a new dimension and "bending down" rays 1, 2, and 3 of the original fan. The morphism $\psi\colon W \to \mathbb{P}^N$ corresponds to the polytope which is a pyramid of height 2 on the polytope of D.



The above polytope corresponds to an embedding of W into $\mathbb{P}^{18}$, with the lattice points corresponding to the 19 homogeneous coordinates. Each homogeneous coordinate is acted on by μ_2, with weight given by the height of the corresponding lattice point. Let x_0, x_a, x_b, and x_c be the homogeneous coordinates corresponding to the labeled lattice points. Following the procedure explained above, we let Y be the intersection of W with the hyperplane defined by $x_0 - x_a - x_b - x_c$.

This Y is a smooth variety with an action of μ_2 so that $X \cong Y/\mu_2$.