Compactifying the universal Picard variety over surfaces

Atoshi Chowdhury

Stanford University

Question

We want to construct a partial compactification of the universal moduli space of line bundles on smooth surfaces, by adding boundary points which parametrize line bundles on reducible surfaces. Ideally, any family of line bundles on smooth surfaces that degenerate to a reducible surface should have a unique limit in our moduli space. What line bundles on reducible surfaces should we include in our space to make this happen?

[image]

Background: the curve case

- The analogous question over curves was answered by Caporaso: the universal Picard variety over M_g is compactified over $\overline{M_g}$ by adding points parametrizing 'balanced' line bundles on 'quasistable' curves.
- The compactification is a GIT quotient of a Hilbert scheme.
- A line bundle L on a nodal curve X is called *balanced* if for any complete subcurve $Y \subset X$,
$$d_Y \geq \frac{d}{g-1}\left(g_Y - 1 + \frac{1}{2}k_Y\right) - \frac{1}{2}k_Y.$$
Here d is the degree of L, g is the genus of X, d_Y is the degree of L restricted to Y, g_Y is the genus of Y, and $k_Y = |Y \cap \overline{X \backslash Y}|$.
- The balanced condition is a stability condition that cuts down on the number of possible limits for a family, thereby combating nonseparatedness.

Nonseparatedness (any dimension)

- Suppose $\mathfrak{X}$ is a smooth one-parameter family of varieties of dimension d whose central fiber X is reducible.
- The irreducible components of X determine line bundles on $\mathfrak{X}$ which are trivial away from X, and twisting any line bundle on $\mathfrak{X}$ by one of these will leave the line bundle unchanged except over X.
- Thus, if a family of line bundles has one line bundle limit over the central fiber, it has infinitely many.

Surfaces: plan of attack

- For a proper moduli space, we want every family of line bundles to have exactly one line bundle limit.
- Given a one-parameter family $\mathfrak{X}$ and a line bundle $\mathcal{L}$ on the generic fiber, does $\mathcal{L}$ have a line bundle limit on the central fiber X? If not, desingularize $\mathfrak{X}$ near X; this guarantees a limit, at the cost of adding exceptional components to X.
- Now we may assume $\mathfrak{X}$ is smooth. If X is reducible, $\mathcal{L}$ will have infinitely many limits (see 'Nonseparatedness').
- We introduce a stability condition, inspired by GIT, generalizing Caporaso's balanced condition for curves. The hope is that only one of the infinitely many limits will be stable.
- The stable limits correspond bijectively to the lattice points in a certain region of $\mathbb{R}^{n-1}$, where n is the number of irreducible components of X. This region is determined by a number of quadratic inequalities that come directly from the stability condition.
- If X has exactly two components, the region turns out to be an interval of length 1, so it contains either one or two lattice points.

Theorem: two-component case

Let $\mathfrak{X}$ be a smooth one-parameter family of surfaces whose central fiber X has two smooth components, Y and Z, which intersect transversely. Assume that the canonical bundle on each component of X is ample. Let $\mathcal{L}$ be a line bundle on $\mathfrak{X}$, let $L = \mathcal{L}|_X$, and let $T_Y = O_{\mathfrak{X}}(Y)|_X$.

If a certain messy expression is not an integer, then there is a unique integer b such that $L \otimes T_Y^b$ is stable. If the expression is an integer, then there are exactly two integers b such that $L \otimes T_Y^b$ is stable.

(Remarks: The values b depend only on X and L, not on $\mathfrak{X}$ and $\mathcal{L}$. I expect to be able to relax the assumption that X has smooth components with ample canonical bundles.)

Sketch of proof

We measure the failure of the stability condition by two quantities $e_Y(b)$ and $e_Z(b)$, quadratic in b (if neither is positive, then L is stable). Calculation shows that $e_Y(b) = -e_Z(b-1)$.

[image]

Stability condition

A line bundle L on a variety X of dimension d is *stable* if for every complete subvariety $Y \subset X$ of dimension d,
$$\frac{h^0(Y,L)}{h^0(X,L)} \geq \frac{1}{L^d \cdot X}\Big(L^d \cdot Y + \frac{1}{d+1}\sum_{j=2}^{d+1}\binom{d+1}{j}(-1)^j D^{j-1}L^{d+1-j}\Big).$$
Here, $Z = \overline{X \backslash Y}$; $D = Y \cap Z$, and multiplication in the sum denotes the intersection product on Z.

If X is the central fiber in a smooth total family $\mathfrak{X}$, the inequality can be written
$$\frac{h^0(Y,L)}{h^0(X,L)} \geq \frac{\frac{1}{d+1}\sum_{j=1}^{d+1}\binom{d+1}{j}(-1)^{j-1}L^{d+1-j}Y^j}{L^d \cdot X}.$$
Here, Y denotes $O_{\mathfrak{X}}(Y)|_X$, and multiplication denotes the intersection product on X.

For $d = 2$ (surfaces), the inequality is
$$\frac{h^0(Y,L)}{h^0(X,L)} \geq \frac{L^2Y - LY^2 + \frac{1}{3}Y^3}{L^2X}.$$

Future questions

- Surfaces with more components (aforementioned region of $\mathbb{R}^{n-1}$ is harder to understand)
- Higher-dimensional varieties (two-component case displays similar phenomena)
- Relationship with GIT
- Extension to vector bundles and relationship to other types of stability

Reference

- L. Caporaso, *A compactification of the universal Picard variety over the moduli space of stable curves*, Journal of the AMS 7, 589-660 (1994).