Objectives:

- Define definite integrals.
- Find areas under curves using definite integrals.

Definitions: If f is a function defined for $a \leq x \leq b$, we divided the interval $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{b-a}{n}.$$

We let $x_0 = a$, x_1, …, $x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any ___sample points___ in these subintervals, so x_i^* is in the ith subinterval $[x_{i-1}, x_i]$. Then the **definite integral of *f* from *a* to *b*** is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)\Delta x$$

provided the limit exists. If the limit does exist, we say that f is ___**integrable** on $[a, b]$___.

Terminology: Let's break down the notation $\int_a^b f(x)\, dx$.

- The symbol $\int$ is called an ___integral sign___
- $f(x)$ is the ___integrand___
- a and b are the ___limits of integration___
- a is the ___lower limit of integration___ and b is the ___upper limit of integration___
- We call computing an integral ___integration___

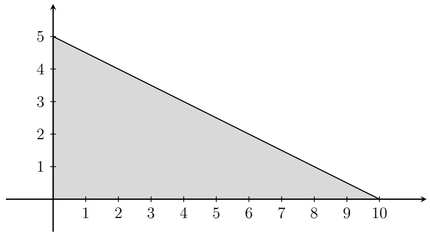
Some intuition: The definite integral is computing ___area between the curve and the x-axis___ but we consider any area above the x-axis is ___positive___ and any area underneath the x-axis is ___negative___.

But wait! Our definition shows that the definite integral is also ___the limit of Riemann sums!___

Some useful things:

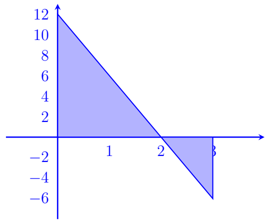
- The sum of the integers from 1 to n: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- The sum of the squares of integers from 1 to n: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- The sum of the cubes of integers from 1 to n: $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$

Example 1 Write down a definite integral that gives the area of the shaded region.



$$\int_0^{10} -\frac{1}{2}x + 5$$

Example 2 Evaluate $\int_0^3 12 - 6t\ dt$ by drawing a the region and computing the area.



Area$=\frac{1}{2}(2)(12) - \frac{1}{2}(1)(6) = 12 - 3 = 9$

Example 3 Evaluate $\int_0^2 \sqrt{4 - x^2}\ dx$ by drawing a the region and computing the area.

$y = \sqrt{4 - x^2}$ is the upper half of the cicle $x^2 + y^2 = 4$, which has center $(0, 0)$ and radius 2. Taking the integral from 0 to 2 gives the area of the right half of this semicirle. $A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(2)^2 = \pi$.

Example 4 A table of values of $f(x)$ is given below. Estimate $\int_0^{12} f(x)\ dx$ using Riemann sums.

x	0	3	6	9	12
$f(x)$	32	22	15	11	9

Right Riemann sum with $n = 4$:

$$3 \cdot 22 + 3 \cdot 15 + 3 \cdot 11 + 3 \cdot 9 = 171.$$

Left Riemann sum with $n = 4$:

$$3 \cdot 32 + 3 \cdot 15 + 3 \cdot 11 = 240.$$

Example 5 Calculate $\int_0^2 x^3\,dx$ exactly using a limit of Riemann sums. We can do this computation in either summation notation or the expanded form.
The right Riemann sum set up: n rectangles; $\Delta x = \frac{2}{n}$; right endpoints: $\frac{2}{n}, \frac{4}{n}, \dots, \frac{2n}{n}$; heights: $\left(\frac{2}{n}\right)^3, \dots, \left(\frac{2n}{n}\right)^3$; summation: $\sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \cdot \frac{2}{n}$. Putting all of this together, we can compute our integral:

$$\begin{aligned}
\int_0^2 x^3\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)^3 \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2^3 i^3}{n^3} \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2^4}{n^4} i^3 \\
&= \lim_{n\to\infty} \frac{2^4}{n^4} \sum_{i=1}^{n} i^3 \\
&= \lim_{n\to\infty} \frac{2^4}{n^4} \left(\frac{n^2(n+1)^2}{4}\right) \\
&= \lim_{n\to\infty} \frac{2^2}{n^2}(n+1)^2 \\
&= \lim_{n\to\infty} 4\frac{n^2+2n+1}{n^2} \\
&= \lim_{n\to\infty} 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \\
&= 4.
\end{aligned}$$

So the area under the curve x^3 between $x = 0$ and $x = 1$ is exactly 4. Cool!

Theorem If $f(x)$ is continuous on $[a, b]$, or if $f(x)$ has only a finite number of jump discontinuities, then f is integrable on $[a, b]$, i.e., the definite integral $\int_a^b f(x)\,dx$ exists.

Things to note: We have assumed that $a < b$ for defining $\int_a^b f(x)\,dx$, but the Riemann sum will allow $a > b$. If $a > b$, then Δx used to be $\frac{b-a}{n}$ and is now $\underline{\Delta x = \frac{a-b}{n}}$. So we have

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

What if $a = b$? Then $\Delta x = \underline{\frac{a-a}{n} = 0}$ so

$$\int_a^a f(x)\,dx = 0$$

Properties of Definite Integrals: Let $f(x)$ and $g(x)$ be continuous functions and c some constant number.

1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Example 6 Evaluate $\int_0^2 (4 + 5x^3)\,dx$.

$$\begin{aligned}\int_0^2 (4 + 5x^3)\,dx &= \int_0^2 4\,dx + \int_0^2 5x^3\,dx \\ &= 4\cdot 2 + 5\int_0^2 x^3\,dx \\ &= 4\cdot 2 + 5\cdot 4 \\ &= 28.\end{aligned}$$