Goal: Use calculus tools on parametric curves to find tangent lines, concavity, and other interesting features.

For the duration of these notes, let $\mathscr{C}$ be a parametric curve given by $x(t)$ and $y(t)$. Let's add a few important assumptions before we start finding tangent lines:

- $x(t)$ is differentiable as a function of t;
- $y(t)$ is differentiable as a function of t;
- For a some interval around our t value of interest, the curve y is differentiable as a function of x. (I like to think about this as playing a .gif instead of the whole movie; maybe my whole movie isn't a function of x but at the small .gif is.)

Let's find the slope of a tangent line at a particular t value, say t_0. That is, we want $\frac{dy}{dx}$ at $t = t_0$.

Near our particular t value, we can treat y as a function of x. That is, $y\big(x(t)\big)$. We can use the chain rule to compute $\frac{dy}{dt}$:

$$\frac{dy}{dt} = \underline{\hspace{3cm}}$$

Solve for $\frac{dy}{dx}$. (You need to impose a condition to keep your equality true. What is this condition?)

$$\frac{dy}{dx} = \underline{\hspace{3cm}}, \text{ provided } \underline{\hspace{3cm}}$$

Woah! The right side doesn't have any occurrence of the parameter t which means we can solve for the slope of a tangent line without eliminating the parameter!

When does $\mathscr{C}$ have a horizontal tangent line?

When does $\mathscr{C}$ have a vertical tangent line?

We can repeat this process to find a formula for our second derivative. You'll end up with this formula:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

1. Let $\mathscr{C}$ be the curve parametrized by

$$\begin{cases} x(t) = t^2 + t, \\ y(t) = t^2 - t, \end{cases} \qquad -2 \leq t \leq 2$$

(a) Sketch the curve.

(b) What is $\dfrac{dy}{dt}$? $\dfrac{dx}{dt}$? $\dfrac{dy}{dx}$?

(c) Find all horizontal and vertical tangent lines of $\mathscr{C}$.

(d) Where is $\mathscr{C}$ concave up? Concave down?

2. Sketch an example of a curve that has a point with multiple tangent lines at a single point. (This example should serve as motivation and a cautionary tale.)

There are two last pieces to the calculus of parametric equations: speed and distance. Suppose $\mathscr{C}$ represents the motion of a particle. The *speed* of the particle traveling along $\mathscr{C}$ at time t is given by

$$\sqrt{\big(x'(t)\big)^2 + \big(y'(t)\big)^2}$$

This formula looks suspiciously like the distance formula but that shouldn't be too surprising since we are asking how distance is changing with respect to t.

That brings us to distance. How can we obtain the total distance traveled by a particle along $\mathscr{C}$ between $t = t_0$ and $t = t_1$? (Hint: We know speed from the formula above.)

2. Here are a few more problems to hone your calculus skills as they relate to parametric equations:

   (i) Consider the parametric curve defined by $\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(2t) \end{cases}$ for $0 \leq t \leq 2\pi$.

   (a) Find $x'(t)$ and $y'(t)$.

   (b) Find $\dfrac{dy}{dx}$.

   (c) Find the tangent line to the curve at $t = \dfrac{\pi}{3}$.

   (d) For what values of t is $y'(t) = 0$? For each of these values, find the point in x-y coordinates.

   (e) For what values of t is $x'(t) = 0$? For each of these values, find the point in x-y coordinates.

   (f) Use technology to graph the curve. For each of the points you found above, describe the tangent line.

   (g) Find the speed of the point $\big(x(t), y(t)\big)$ as it moves along the curve at $t = 0$.

(ii) Consider the parametric curve defined by $\langle e^t - t, t^2 - t\rangle$ for $-1 \le t \le 1$.

(a) For what points on the curve is the tangent line horizontal?

(b) For what points on the curve is the tangent line vertical?

(iii) Consider the parametric curve defined by $\begin{cases} x(t) = \sin(t) \\ y(t) = \left(\dfrac{t}{\pi}\right)^2 \end{cases}$ for $-\dfrac{3}{2}\pi \le t \le \dfrac{3}{2}\pi$.

(a) For what values of t does the curve intersect itself? (Hint: If $\big(x(t), y(t)\big) = \big(x(s), y(s)\big)$, then $x(t) = x(s)$ and $y(t) = y(s)$. We know $x(t) = x(s)$ if $t = s + 2\pi$, so try solving $y(s + 2\pi) = y(s)$ for a value of s.)

(b) For one of the values of t above, find the tangent line to the curve. Then find the tangent line for the other value of t.

(c) Use technology to help you sketch the curve. Draw both tangent lines on the graph.

(iv) For each part, draw a curve that intersects itself in such a way that:

(a) the two tangent lines at the intersection are perpendicular.

(b) for both values of t at the intersection, the tangent line is the same.

(c) the two tangent lines at the intersection are neither the same, nor perpendicular.

(v) Integrating speed gives us arc length. For functions $f(x)$, $g(y)$

(a) Write $f(x)$ and $g(y)$ as parametric equations.

(b) Use these parametric equations and the equation for speed to arrive at the formulas for (1) the arc length of $f(x)$ from $x = a$ to $x = b$, and (2) the arc length of $g(y)$ from $y = c$ to $y = d$.