**Goal:** Compute areas and arc length in polar coordinates.

## Arc Length

Suppose a curve $\mathscr{C}$ is given by parametric equation $x(t)$ and $y(t)$. What is the formula for the arc length of the curve between $t = a$ and $t = b$?

Let's consider a polar curve $r = f(\theta)$, $a \leq \theta \leq b$. We can consider $\theta$ as our parameter. How can we set up our parametric equations for $x$ and $y$ in terms of just $\theta$?

$x =$ $y =$

In hopes of substituting into the arc length formula, differentiate both equations above (with respect to $\theta$) using the product rule to obtain $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$.

Now, compute the expression that lives inside the square root in our arc length formula.

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 =$$

Assuming that $f'$ is continuous, we can now put all the pieces together. The length of a curve with polar equation $r = f(\theta)$, $a \leq \theta \leq b$ is

$$L =$$

**Area**

Let's think way back to how we defined the area under a curve in Calculus I. We took little slices of $x$-change and estimated the area under the curve with a rectangle using the area formula

$$A = (\text{length})(\text{width}) = f(x) \cdot \Delta x.$$

Next, we took smaller and smaller $\Delta x$ to get an integral! We're going to do the same thing but for polar coordinates.

First, if we change $\theta$ a little bit, we end up tracing out a sector of a circle (as opposed to a rectangle). The area of a sector of a circle of radius $r$ and central angle $\theta$ (measured in radians) is given by

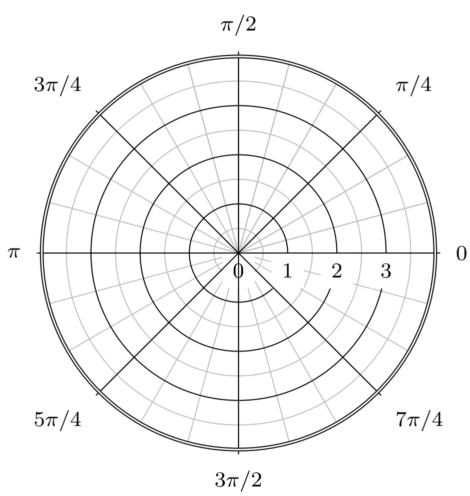
$$A = \frac{1}{2} r^2 \Delta\theta.$$

Your text walks through the full Riemann sum description but we'll jump right to the end. Let $\mathscr{R}$ be the region bounded by the polar curve $r = f(\theta)$ and the rays $\theta = a$ and $\theta = b$, where $f$ is a positive and continuous function and $0 < b - a < 2\pi$. The area of the polar region $\mathscr{R}$ is given by

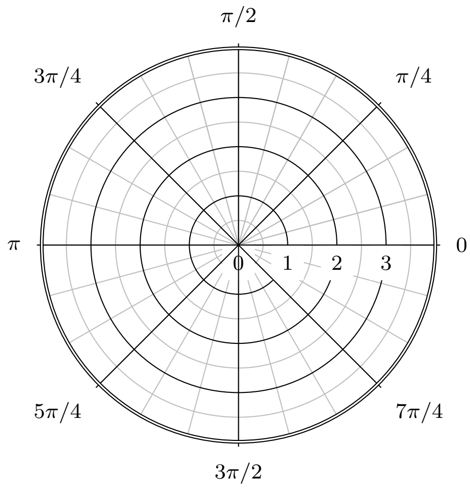
$A =$

We can also find the areas between polar curves, but finding intersections can be tricky so remember, *always draw a picture!*

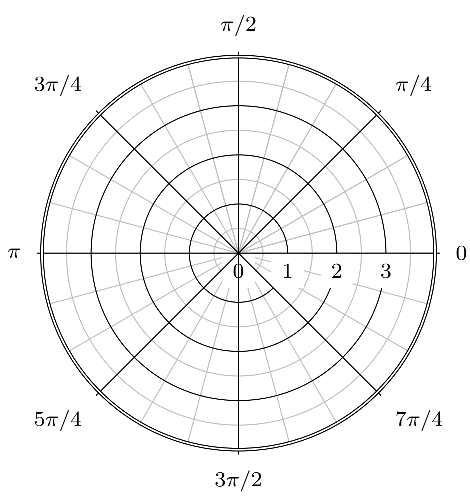
1. Find the length of the cardioid $r = 1 + \sin\theta$.



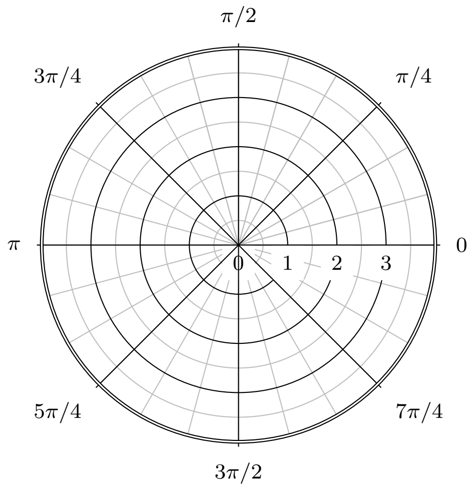
2. Find the area of the region that is bounded by $r = \theta^2$, $0 \le \theta \le \pi/4$.




3. Find the length of the curve $r = \theta^2$, $0 \le \theta \le 2\pi$.

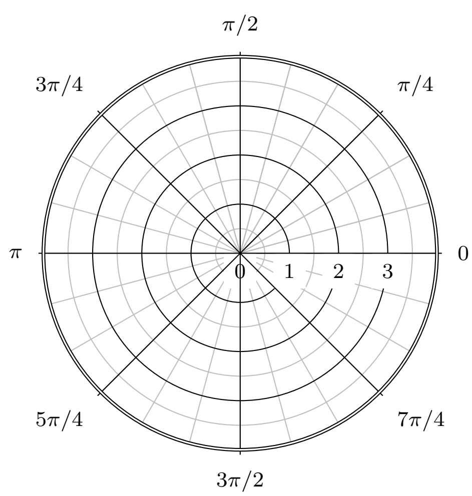


4. Find the area of the region enclosed by one loop of the curve $r = \cos(2\theta)$.




5. Here are a few more area and arc length questions for you to show your mastery!

   (i) Find the area inside the region bounded by $r = 3 + 3\sin\theta$.

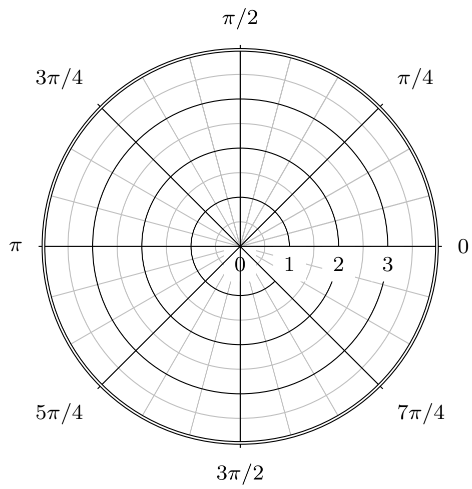


(ii) Consider the curve $r = 2\csc\theta$.

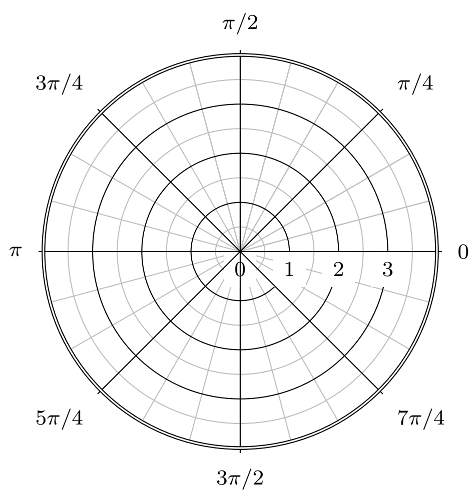
(a) Convert this curve into rectangular coordinates. This will give us a parametrization with $\theta$ serving the role of $t$.

(b) What is the slope of the curve at any point $(x, y)$?

(c) Find the length of the curve from $\theta = \pi/6$ to $\theta = \pi/2$.




(iii) Find the arc length of the cardioid $r = 3 + 3\sin\theta$.



(iv) ☕ Consider the curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$.

(a) Sketch the curves.

(b) Find the values of $\theta$ where the curves intersect.

(c) Find the area of the region inside both curves.

(d) Find the area of the region inside $r = 3\sin\theta$ and outside $r = 1 + \sin\theta$.