Goal: Play with Taylor series!

1. We can use series to evaluate limits! Evaluate $\lim_{x \to 0} \dfrac{x - \ln(1+x)}{x^2}$.

   (a) Find a series representation for $\dfrac{x - \ln(1+x)}{x^2}$.

   (b) Take the limit!

2. We can find values of series! Find the sums of the following series.

   (a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n3^n}{5^n}$

3. We can use Taylor series to approximate integrals! Suppose we want to estimate $\int_0^1 x\cos(x^3)\,dx$ to within three decimal places ($|\text{error}| \leq 0.0005$).

   (a) Use the Maclaurin series $\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ to find the Maclaurin series for $f(x) = x\cos(x^3)$.

   (b) Find the general antiderivative $F(x) = \int x\cos(x^3)\,dx$ by integrating the Maclaurin series for $f(x)$.

   (c) Remember $\int_0^x f(t)\,dt = F(x) - F(0)$. Show $F(0) = 0$. Then $\int_0^1 f(t)\,dt = F(1)$.

   (d) We have a Maclaurin series for $F(x)$. Using this series, what is $F^{(n+1)}(x)$ for a given n?

   (e) What is the maximum value of $|F^{(n+1)}(x)|$ for $|x - 0| \leq 1$?

   (f) Use Taylor's Inequality with the value of M you just found to find the number of terms necessary for $|R_n(1)| \leq 0.0005$. (Guess-and-check or graph.)

   (g) Find the partial sum for $F(1)$ for the value of n you just found. This is an estimate for $\int_0^1 x\cos(x^3)\,dx$ to within three decimal places.

4. Here are all sorts of problems related to Taylor series. Soon you will be masters!

   (i) ♨ Use series to approximate the definite integral $\int_0^{0.2} \arctan(x^3) + \sin(x^3)\, dx$ to within five decimal places.

   (ii) ♨ (Must do all parts) Find the sums of the series below:

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!}$

   (b) $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

   (c) $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots$

(iii) ♨ Prove Taylor's inequality for $n = 2$, that is, prove that if $|f'''(x)| \leq M$ for $|x-a| \leq d$, then

$$|R_2(x)| \leq \frac{M}{6}|x-a|^3, \quad \text{for } |x-a| \leq d$$

(iv) ♨ Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series. Graph the function and comment on it's behavior near the origin.