Goal: Estimate the error when using a Taylor polynomial to approximate a function. Determine when a function f has a representation as a Taylor series.

How can we show that any function *actually* has a power series representation?

We need to show that $f(x)$ is the limit of the partial sums of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ (that is, show the series converges by the definition and show that it converges to f).

What is the nth partial sum of the Taylor series for f?

$T_n(x) =$

Recall that this is called the nth-degree Taylor polynomial of f centered at a. So $f(x)$ is equal to its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

We can define the *remainder* $R_n(x)$ of the Taylor series as follows:

$$R_n(x) = f(x) - T_n(x), \quad \text{so} \quad f(x) = \underline{\hspace{3cm}}$$

We need the following chain of equalities to hold:

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}\big(f(x) - R_n(x)\big) = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

What requirement can we place on $\displaystyle\lim_{n\to\infty} R_n(x)$ to make the above equalities hold?

$\displaystyle\lim_{n\to\infty} R_n(x) =$

We've just proved another theorem!

Theorem: If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth degree Taylor polynomial of f at a and

$\displaystyle\lim_{n\to\infty} R_n(x) =$

for $|x-a| < R$, then f is equal to the sum of it's Taylor series on the interval $|x-a| < R$.

So that leaves the question: How can we show that $\displaystyle\lim_{n\to\infty} R_n(x) = 0$ for a specific function f?

Taylor's Inequality: If $\left|f^{(n+1)}(x)\right| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \leq d$$

Alternate statement of Taylor's Inequality: If $\left|f^{(n+1)}(x)\right| \leq M$ between a and x, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

In general, this isn't too hard to prove. The text outlines a really nice proof when $n = 1$ on page 607 and for your portfolio, you can show that the statement is true for $n = 2$.

Let's see Taylor's inequality in action with one of our favorite Taylor series!

1. Prove that e^x is equal to the sum of its Maclaurin series for all x.

   (a) What is the Maclaurin series for e^x?

   $$e^x =$$

   (b) What is the $(n+1)$st derivative of $f(x) = e^x$?

   $$f^{(n+1)}(x) =$$

   (c) Let d be any positive real number such that $|x| \leq d$. What is an upper bound for $\left|f^{(n+1)}(x)\right|$? (That is, choose M. Note that M can depend on d, but it must be independent of n.)

   $$\left|f^{(n+1)}(x)\right| \leq$$

   (d) Fill in the values in Taylor's inequality.

   $$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} =$$

   (e) Now, take a limit of your upper bound!

   $$\lim_{n\to\infty} \frac{M}{(n+1)!}|x-a|^{n+1} =$$

   (f) Which theorem allows you to conclude $\lim\limits_{n\to\infty} |R_n(x)| = 0$?

   (g) Conclude that e^x is equal to the sum of its Maclaurin series.

   (h) Give a expression in terms of a series for e^π.

2. Show that $\sin(x)$ is equal to it's Maclaurin series for all x.

3. Let $f(x) = \sin(x)$

   (a) State that the Maclaurin series for $f(x)$. For what values of x does the Maclaurin series converge?

   (b) Suppose we want to use a partial sum of the Maclaurin series to estimate $\sin(\pi/3)$. What is $T_5(x)$?

   (c) What is $f^{(6)}(x)$?

   (d) Recall the Maclaurin series is a Taylor series centered at $a = 0$. If $|x - 0| \leq \pi/3$, what is the maximum value of $|f^{(6)}(x)|$?

   (e) In Taylor's inequality, the maximum found in part (d) is called M. So we have,

$$|R_5(\pi/3)| \leq \frac{M}{(5+1)!}|\pi/3 - 0|^{5+1}$$

   Plug in your value for M and simplify to find an upper bound for $|R_5(\pi/3)| = |T_5(\pi/3) - \sin(\pi/3)|$.