Goal: Define the Taylor series of a function f.

Suppose that $f(x)$ has a power series representation centered at $x = a$ with radius of convergence R:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n, \qquad |x-a| < R$$

You've already spent some time working with the expressions these coefficients must have, but today, we are going to spend some time demystifying them. First, plug $x = a$ into the equation above. What is $f(a)$?

$f(a) =$

We know from the previous section that we can differentiate a function by differentiating the power series representation term by term. What is $f'(x)$ as a series?

$f'(x) =$

Let's plug $x = a$ into this new equation. What is $f'(a)$?

$f'(a) =$

We seem to be making progress! Let's do it again! What is $f''(x)$ as a series? What is $f''(a)$?

$f''(x) =$

$f''(a) =$

What about the nth derivative? If we take the nth derivative of $f(x)$, what series do we obtain? What happens when we plug in $x = a$?

$f^{(n)}(x) =$

$f^{(n)}(a) =$

Woah! We've proved a theorem!

Theorem: If f has a power series representation at a, that is, if $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for $|x-a| < R$, then its coefficients are given by the formula

$$c_n =$$

This fact motivates the following definitions.

The Taylor series for $f(x)$ centered at $x = a$ is given by

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We often choose to center our Taylor series for a function at $x = 0$. We call this *the Maclaurin series for $f(x)$* and it is given by

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Recap: We know that *if* a function has a power series representation centered at $x = a$, that must be the Taylor series. We will work on showing specific functions really do have power series representations soon. Stay tuned!

1. If $f(x) = e^x$, find the Maclaurin series and its radius of convergence.

   At this point, the only conclusion we can make is that *if* e^x has a power series representation at $x = 0$, then

$$e^x = \sum_{n=0}^{\infty}$$

2. Find the Taylor series for $f(x) = e^x$ centered at $a = 3$. What is the radius of convergence?

3. Find the Maclaurin series for $f(x) = \sin(x)$. What is the radius of convergence?

4. Find the Maclaurin series for $g(x) = \cos(x)$ and the radius of convergence. (You can do this directly but you may find the previous question useful.)

5. Find the Taylor series for $f(x) = \dfrac{1}{1-x}$ about $a = 0$ and the radius of convergence.

6. Now that you've got some Taylor series experience, it's time to become an expert! For each function below, find the Taylor series centered at a. Give the radius of convergence.

   (i) $g(x) = \arctan(x)$, $a = 0$

   (ii) $f(t) = \ln(1 + x)$, $a = 0$

(iii) $f(x) = x - x^3$, $a = -2$

(iv) ☕ $f(x) = \cos(x)$, $a = \pi$

(v) ☕ $f(x) = x^{-2}$, $a = 1$