Goal: Define a Taylor polynomial for a function f and use this polynomial to approximate f.

In Calculus I, you learned how to approximate function values using lines (linear approximation). Let's start there.

1. Approximate $e^{0.1}$ using only a four-function calculator and your knowledge of calculus.

   (a) Use the tangent line approximation with $f(x) = e^x$. Compare to the value given by your calculator. (Recall that the formula for a tangent line approximation for $f(x)$ near $x = a$ is given by $f(x) \approx L(x) = f(a) + f'(a)(x - a)$.)

   (b) In part (a), we only used the first derivative to approximate $e^{0.1}$. We can get a better estimate by using the second derivative. Write down a polynomial that will match f, f', and f'' at $x = a$. Use this to approximate $e^{0.1}$ and compare this value with the value given by your calculator.

   (c) What properties of f are we modeling with our approximating line and our approximating parabola?

   (d) As masters of calculus, how could we get an even better estimate?

The nth degree Taylor polynomial for $f(x)$ centered at $x = a$ is given by

$$\begin{aligned} T_n(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \end{aligned}$$

2. Consider $f(x) = e^{3x}$.

   (a) Write out the fifth degree Taylor polynomial $T_5(x)$ for $f(x) = e^{3x}$ about $x = 0$. Express your answer in expanded form and in summation notation.

   (b) Use your polynomial to approximate $e^{1.5}$ and evaluate using technology. (This expression is called the 6th partial sum approximation for $e^{1.5}$.)

   (c) Express the 20th partial sum approximation for $e^{1.5}$ using summation notation and evaluate using technology.

   (d) Express an exact series representation for $e^{1.5}$ and justify why the series you wrote should converge to $e^{1.5}$.

   (e) Conjecture an exact series representation in summation notation for e^{3x}. Is this series a power series? What is the interval of convergence?

3. Consider $f(x) = \dfrac{1}{x^2}$.

   (a) Write out the fifth degree Taylor polynomial $T_5(x)$ for $f(x) = \dfrac{1}{x^2}$ about $x = 1$. Express your answer in expanded form and in summation notation.

   (b) Use your polynomial to approximate $\dfrac{1}{(1.2)^2}$ and evaluate using technology.

   (c) Express the 20th partial sum approximation for $\dfrac{1}{(1.2)^2}$ using summation notation and evaluate using technology.

   (d) Express an exact series representation for $\dfrac{1}{(1.2)^2}$ and justify why the series you wrote should converge to $\dfrac{1}{(1.2)^2}$.

   (e) Conjecture an exact series representation in summation notation for $\dfrac{1}{x^2}$. Is this series a power series? What is the interval of convergence?

4. The function $f(x)$ is approximated near $x = 3$ by the 4th degree Taylor polynomial

$$T_4(x) = 1 + 5(x-3) - 2(x-3)^2 + 9(x-3)^3 - \frac{1}{6}(x-3)^4$$

What are $f(3)$, $f'(3)$, $f''(3)$, $f'''(3)$, and $f^{(4)}(3)$?