Goal: Formalize how we can represent familiar functions as power series and build new power series representations for functions by integrating or differentiating a known power series.

Theorem: If the power series $\sum c_n(x-a)^n$ has radius of converge $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

is differentiable (and therefore continuous) on $(a-R, a+R)$ and

(i) $f'(x) =$

(ii) $\displaystyle\int f(x)\,dx =$

The radii of convergence of the power series in (i) and (ii) are both R.

1. Let's build a power series for $f(x) = \dfrac{x^2}{(1+x)^3}$.

   (a) What is the antiderivative of $g(x) = \dfrac{1}{(1+x)^2}$? Give it's power series and radius of convergence.

   (b) Use part (i) to find a power series for $h(x) = \dfrac{1}{(1+x)^2}$.

   (c) Use part (ii) to find a power series for $k(x) = \dfrac{1}{(1+x)^3}$.

   (d) Use part (iii) to find a power series for $f(x) = \dfrac{x^2}{(1+x)^3}$. What is the radius of convergence?t

2. Our goal is to find a power series representation for $f(x) = \dfrac{1+x}{1-x}$.

   (a) Find the power series for $g(x) = \dfrac{1}{1-x}$ and its radius of convergence.

   (b) Use part (a) to find a power series representation for $h(x) = \dfrac{x}{1-x}$.

   (c) We want to add the two series together to get a series representation of $f(x)$. This will be easier if both series have terms with the same power of x. Use an index shift to rewrite the series for $h(x)$ so that its terms include x^n.

   (d) Now we want to add the two series together. Try writing each series in expanded form and then writing a new series expression for the sum.

   (e) What is the radius of convergence of the final series for $\dfrac{1+x}{1-x}$?

3. Now that you have a whole new way to think about functions, try these problems on your own!

   (i) Find a power series representation for $f(x) = \dfrac{3}{1-x^4}$. What is the radius of convergence?

   (ii) Find a power series representation for $g(x) = \dfrac{x}{9+x^2}$.

   Hint: Find a representation for $\dfrac{1}{1+(x^2/9)}$ first.

   (iii) Give the antiderivative of $\arctan(x^2)$ as a power series.

(iv) Find a power series representation for the function $h(x) = \dfrac{x^3}{(1+5x)^2}$. What is the radius of convergence?

(v) ☕ Find a power series representation for the function $g(x) = \left(\dfrac{x^2}{7-x}\right)^3$. What is the radius of convergence?

(vi) ☕ Find a power series representation for $f(x) = \dfrac{x^2}{a^4 - x^4}$ and give the radius of convergence.