Goal: Investigate and understand functions using a series – specifically power series.

Let's jump right in. Below are a few functions defined by series.

1. Consider $f(x) = \sum_{k=0}^{\infty} k^2 x^k$.

   (a) Write the series $f(x)$ should give for $x = 1$, $x = \frac{1}{2}$, $x = -2$, and $x = 0$.

   (b) For which of these inputs is $f(x)$ defined?

   (c) For which values of x do you think $f(x)$ is defined?

2. Consider $g(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x}{2n}$.

   (a) Write the series $g(x)$ should give for $x = 1$, $x = \frac{1}{2}$, $x = -2$, and $x = 0$.

   (b) For which of these inputs is $g(x)$ defined?

   (c) For which values of x do you think $g(x)$ is defined?

3. Consider $h(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(x-1)^k}{3^k}$.

   (a) Write the series $h(x)$ should give for $x = 1$, $x = \frac{1}{2}$, $x = -2$, and $x = 0$.

   (b) For which of these inputs is $h(x)$ defined? What will the associated output value be?

   (c) For which values of x do you think $h(x)$ is defined? What will the associated output value be?

4. Consider $\ell(x) = \displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{2}\right) x^n$.

   (a) Write the series $\ell(x)$ should give for $x = 1$, $x = \frac{1}{2}$, $x = -2$, and $x = 0$.

   (b) For which of these inputs is $\ell(x)$ defined? What will the associated output value be?

   (c) For which values of x do you think $\ell(x)$ is defined? What will the associated output value be?

5. Consider $p(x) = \sum_{n=1}^{\infty} 7\left(x - \frac{1}{n^2}\right)$.

   (a) Write the series $p(x)$ should give for $x = 1$, $x = \frac{1}{2}$, $x = -2$, and $x = 0$.

   (b) For which of these inputs is $p(x)$ defined?

   (c) For which values of x do you think $p(x)$ is defined?

A *power series about* $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

where x is a variable and the c_n's are constants (called the *coefficients* of the series). We call $x = a$ the *center* of the power series.

Which of the series above are power series? For those series, identify the coefficients c_n and the center $x = a$.

The domain of a power series (the values for which the series that defines the functions converges) is called the *interval of convergence* for that power series. For each of the power series below, find the interval of convergence Remember to check endpoints of your intervals which often correspond to inconclusive test results.

6. $\displaystyle\sum_{n=0}^{\infty} x^n$

7. $\displaystyle\sum_{k=1}^{\infty} n!x^n$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$

9. $J_0(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$ (called the *Bessel function*)

Usually, we will use the ratio test to determine an interval of convergence for a power series and the possible results are summarized in the following theorem:

Theorem: For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(i) The series converges for all x.

(i) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R is called the *radius of convergence* of the power series and we make two conventions: if the series converges only for $x = a$, then $R = 0$, and if the series converges for all x, then $R = \infty$.

10. Now that you've gotten a taste for power series, find the radius of convergence and interval of convergence for each of the following power series.

    (i) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{x}}$

    (ii) $\displaystyle\sum_{n=0}^{1} \frac{x^n}{n!}$

(iii) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$

(iv) $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

(v) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

(vi) If $\displaystyle\sum_{n=0}^{\infty} c_n 4^n$ is convergent, is $\displaystyle\sum_{n=0}^{\infty} c_n(-2)^n$ convergent? Is $\displaystyle\sum_{n=0}^{\infty} c_n(-4)^n$ convergent? Fully justify your claims.

(vii) If $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ where $c_{n+4} = c_n$ for all $n \geq 0$, find the interval of convergence of the series and a formula for $f(x)$.