Goal: Understand different types of convergence. Use the ratio test to determine if a series converges absolutely.

Can you think of an example of a series b_n of positive terms such that $\sum_{n=1}^{\infty}(-1)^n b_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges?

We say that a series $\sum a_n$ is *absolutely convergent* if the series of absolute values $\sum |a_n|$ converges. This is the strongest notion of convergence that we have! If a series is convergent but not absolutely convergent, we say the series is *conditionally convergent*.

Fact: If a series $\sum a_n$ is absolutely convergent, then it is convergent. Let's prove this.

(a) Since that $|a_n|$ is either equal to a_n or $-a_n$, we have the inequality

$$0 \leq a_n + |a_n| \leq 2\,|a_n|$$

Use the comparison test to show that $\sum_{n=1}^{\infty} a_n + |a_n|$ is convergent.

(b) Use the result of (a) to show that $\sum_{n=1}^{\infty} a_n$ is convergent.

Comment: What can we do if our series doesn't have exclusively positive terms and also isn't alternating? We hope that the series is absolutely convergent and we hit it with any test we can (including the one below) but if that doesn't work or shows the series of absolute values diverges, we really only have the sequence of partial sums to use for showing conditional convergence.

1. Determine if $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ is absolutely convergent, conditionally convergent, or divergent.

When looking for absolute convergence, let's add one more test to our toolbox that will undoubtedly come in handy.

Ratio Test: Let $\sum a_n$ be a series. Define

$$L =$$

- If $L < 1$, the series is ________________________________
- If $L > 1$ or the limit is infinite, the series is ________________________________
- If $L = 1$, ________________________________

2. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ is absolutely convergent, conditionally convergent, or divergent.

3. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$ is absolutely convergent, conditionally convergent, or divergent.

4. Here are some more problems to work on your new tool: the ratio test!

   (i) Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{(n+2)5^{2n+1}}$ is absolutely convergent, conditionally convergent, or divergent.

   (ii) Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n)!}$ is absolutely convergent, conditionally convergent, or divergent.

   (iii) For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$