Goal: Estimating a series using only finitely many terms, we can bound our error. Also, given a level of precision, we can determine the number of terms we should use in our estimate to guarantee the appropriate level of accuracy.

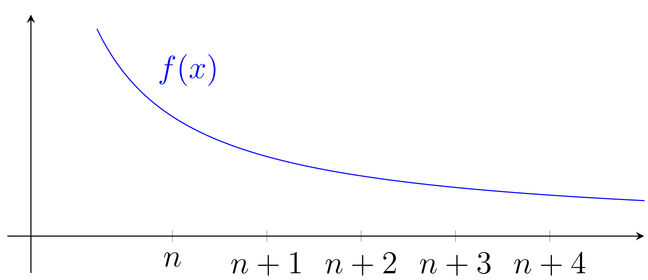
Suppose we have a series $\sum a_k$ that converges by the integral test. Write out the integral test below and underline the hypotheses:

We want to estimate the value of this series using a partial sum. For any positive integer n, write out the partial sum s_n below:

How good of an approximation is s_n? We need to estimate the size of the remainder:

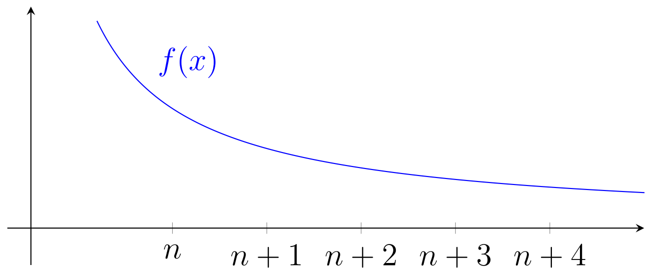
$$R_n = \left(\sum_{k=1}^{\infty} a_k\right) - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

On the figure below, let $f(x)$ be the function in the integral test that agrees with a_n on the integers. Draw three rectangles on the graph which visually represent the sequence elements a_{n+1}, a_{n+2}, and a_{n+3} and which correspond to a right-hand Riemann sum approximation (that is, the rectangle between n and $n+1$ should have height $f(n+1) = a_{n+1}$).



Using an integral, what is an upper bound for R_n?

Using the same figure as above, draw three rectangles on the graph which visually represent the sequence elements a_{n+1}, a_{n+2}, and a_{n+3} and which correspond to a left-hand Riemann sum approximation (that is, the rectangle between $n+1$ and $n+2$ should have height $f(n+1) = a_{n+1}$).



Using an integral, what is a lower bound for R_n?

Remainder Estimate for the Integral Test: Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n = s$ is convergent. If $R_n = s - s_n$, then

$$\underline{\hspace{4cm}} \leq R_n \leq \underline{\hspace{4cm}}$$

In the inequality above, replace R_n with $s - s_n$.

Add s_n to all parts to express an interval in which the true value of the series must lie.

1. How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ do we need to add to estimate the true value of the series correct to within 3 decimal places?

Suppose we have a series $\sum(-1)^{k-1}b_k$ that converges by the alternating series test. Write out the alternating series test below and underline the hypotheses:

We want to estimate the value of this series using a partial sum. For any positive integer n, write out the partial sum s_n below:

Based on your intuition, can you bound the remainder from above?

Remainder Estimate for Alternating Series: If $s = \sum(-1)^{n-1}b_n$ is the sum of an alternating series that satisfies $b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then

$$R_n = |s - s_n| \leq \underline{\qquad\qquad}$$

2. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{n!}$ correct to two decimal places.

3. To sharpen your series estimation skills, here are some problems to work on!

   (i) Use the sum of the first 10 terms to estimate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. How good is this estimate? Using this estimate, give an interval in which s must lie. Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

   (ii) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n5^n}$ is convergent. How many terms of the series do we need to add in order to find the sum to the accuracy $|\text{error}| < 0.0001$?

   (iii) How many terms of the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ would you need to add to find its sum to within 0.01?

(iv) Is the 103rd partial sum s_{103} of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ an over estimate or an underestimate of the total sum. Explain.

(v) Estimate $\sum_{n=1}^{\infty}(2n+1)^{-6}$ correct to five decimal places.

(vi) Use $s_1 6$ to estimate $\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2+1}$ and estimate the error.

(vii) Show that $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{8^n}$ converges. Estimate the sum correct to four decimal places.

(viii) Use the sum of the first 10 terms to approximate the sum of the series $\sum_{n=1}^{\infty}\frac{\big(\sin(n)\big)^2}{n^3}$. Estimate the error. Using this estimate (s_{10}), give an interval in which the true value must lie.