Goal: Define series and what it means for a series to converge or diverge along with use a test for divergence.

A *series* is an attempt to add up infinitely many terms. I say "attempt" because it is often the case that this doesn't make sense but more on that later. Let's look at an example that you've seen before. Let's consider the decimal

$$S = 1.\bar{1} = 1.11111\ldots$$

You might recognize $S = 1 + \frac{1}{9}$ which is a finite sum. But what do we mean when we write S as a repeating decimal? Since we use the base 10 system, we can fill in the numerators below.

$$S = 1 + \frac{\quad}{10} + \frac{\quad}{10^2} + \frac{\quad}{10^3} + \frac{\quad}{10^4} + \frac{\quad}{10^5} + \cdots$$

The three dots ($\cdots$) means that the sum continues forever. The more terms we add, the closer we get to S.

Let's build some notation. Let $\{a_n\}_1^\infty$ be a sequence. The associated series is denoted in any of the following ways:

$$\sum_{n=1}^{\infty} a_n, \qquad \sum a_n, \qquad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

The symbol Σ is a capital sigma but we will usually say "sum" and we call notations involving Σ "summation notation" or "Sigma notation."

Write the series for S in sigma notation.

$S =$

Does it always make sense to talk about a sum of infinitely many terms? Think about the series

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots + n + \cdots$$

Let's look at the sum of the first n terms. What does that equal? (If you don't know, Google it! It may shock you that there's such a nice formula for the sum of the first n positive integers but there is a neat explanation!)

$1 + 2 + 3 + \cdots + n =$

As n gets large, this expression gets huge! Do you think this infinite sum is really a finite number?

Let's gather some definitions that will help us make this concept more concrete. Consider the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

- Define an associated sequence, s_n of the nth partial sums:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

- If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists (as a real number), then we say the series $\sum a_n$ *converges* and we write

$$s = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

- We say that the number s is the sum of the series.
- If the sequence $\{s_n\}$ diverges, we say the series $\sum a_n$ *diverges.*

So for any series $\sum a_n$, there are two associated sequences: the sequence of terms $\{a_n\}$ and the sequence of partial sums $\{s_n\}$. Plus they all have limits! It can get really confusing so develop a method ASAP that will help you keep them straight.

Also, since there are so many pieces floating around, you have to be very clear in your solutions which piece you are working with. "The sequence," "the limit," "it converges," and "it diverges" are all really horrible for distinguishing what you're working with and why it might matter. Goal: Be annoyingly specific!

Generally, we're going to work on gaining tools that will allow us to determine if a series converges or diverges without needing to work directly with the sequence of partial sums but for now, that's all we really have so let's get to it!

1. Consider the sequence $a_n = \dfrac{1}{n(n+1)}$. Show the associated series converges and find its sum.

   (a) Write the expanded definition of the nth partial sum

   $$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} =$$

   (b) To understand the sequence of partial sums better, use partial fraction decomposition to separate $\dfrac{1}{i(i+1)}$.

   (c) Rewrite the expanded version of s_n using this partial fraction decomposition. All but two terms should cancel.

   $$s_n = \sum_{i=1}^{n} \left(\frac{}{i} + \frac{}{i+1} \right) =$$

   (d) Using this expression, what is $\lim_{n\to\infty} s_n$?

   (e) What is $\sum_{n=1}^{\infty} a_n$?

2. Consider the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots$$

(The pattern that continues is that each term $\frac{1}{k}$ appears k times.)

(a) Find the first 10 terms in the sequence of partial sums.

(b) Using a pattern you see, convince me (using words rather than indices and formulas) that for any number M, there exists some partial sum that exceeds M.

(c) Does this series converge or diverge?

(d) What is the limit of the original sequence given (the individual terms we are adding)? Is this surprising?

Divergence Test: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n =$ _____.

More Useful Restatement of the Divergence Test: Given a sequence $\{a_n\}$, if $\lim_{n\to\infty} a_n$ is not 0, then ________________________.

Warning: If $\lim_{n\to\infty} a_n = 0$, we can NOT say that $\sum a_n$ converges! Beware this tempting trap! Knowing $\lim_{n\to\infty} a_n = 0$ doesn't give us any information about the convergence or divergence of the associated series.

3. Let $a_n = \dfrac{2n}{3n+1}$.

   (a) Determine if $\{a_n\}$ is convergent and if it converges, what it converges to.

   (b) What is the associated series?

   (c) Determine if the associated series is convergent.

4. Determine if $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$ converges or diverges.

5. Consider the *harmonic series*: $\sum_{n=1}^{\infty} \frac{1}{n}$. This series is quite famous and should constantly be an example in your pocket of a series that diverges but the terms of the associated sequence $a_n = \frac{1}{n}$ converges to zero. We will go through this in class but it is also done on page 569.

Theorem: If $\sum a_n$ and $\sum b_n$ are convergent series and c is a constant, then the following series also converge and you should fill in what they converge to on the right hand side (pg. 570).

- $\sum_{n=1}^{\infty} c a_n =$
- $\sum_{n=1}^{\infty} (a_n + b_n) =$
- $\sum_{n=1}^{\infty} (a_n - b_n) =$