Goal: Define sequences and determine whether a sequence converges or diverges.

There are a few ways we can define a sequence. My favorite: A *sequence* is a function whose domain is the natural numbers (or a subset but let's think about that later) and whose output (for any particular input) can any real number. More casually, a sequence is a list of numbers written in a definite order:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

where we say a_1 is the first term, a_2 is the second term, and in general, a_n is the general term. For this class, we will almost always be talking about infinite series so a_n will always be followed by a_{n+1}.

Write three sequences. Any three will do!

1.

2.

3.

We can denote sequences with various notations, the most common (other than the above) are

$$\{a_1, a_2, a_3, \ldots\}, \quad \{a_n\}_1^\infty.$$

Sometimes you'll see a sequence described by just the nth term a_n but this is generally bad practice since a_n is a number while $\{a_n\}$ implies a list of numbers.

Write our the first five terms of the following sequences described by their general term.

1. $\left\{\dfrac{n}{n+1}\right\}_1^\infty$

2. $\sqrt{n-3},\ n \geq 3$

3. $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}_1^\infty$

Find a formula for the general term a_n of the following sequences assuming that the pattern of the first few terms continues.

1. $\left\{\dfrac{3}{5}, -\dfrac{4}{25}, \dfrac{5}{125}, -\dfrac{6}{625}, \dfrac{7}{3125}, \ldots\right\}$

2. $\{5, 1, 5, 1, 5, 1, \ldots\}$

Let's define some terms related to sequences.

- A sequence $\{a_n\}$ has the *limit* L if we can make the terms a_n as close to L as we like by taking n sufficiently large. We write
$$\lim_{n\to\infty} a_n = L.$$
- If $\lim_{n\to\infty} a_n$ exists, we say the sequence *converges*. Otherwise, we say the sequence *diverges*.

Let's talk about limits a little more. In your previous calculus experience, you've only taken limits of functions defined over some interval of the real line. Now, we need to take limits of things defined over just the natural numbers (i.e., a discrete subset of the real line). We can get a better handle on this by finding a function (over $\mathbb{R}$ that matches our sequence on the natural numbers so we can use our old limit tools. Formally, we can write this idea as a theorem.

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is a natural number, then $\lim_{n\to\infty} a_n = L$.

Usually, we can do this by replacing n with x in our general term but you have to be careful. Also, this might seem a bit pedantic, but what would it mean to use l'Hôpital's rule on a sequence? This swap to a function allows that to make sense.

Here are the limit laws for sequences. Feel free to use your experience with the limit laws for functions and guess what each law should be and check with me or the book (pg. 557) to make sure you've got it.

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

- $\lim_{n\to\infty} (a_n + b_n) =$
- $\lim_{n\to\infty} (a_n - b_n) =$
- $\lim_{n\to\infty} ca_n =$
- $\lim_{n\to\infty} (a_n b_n) =$
- $\lim_{n\to\infty} \frac{a_n}{b_n} =$
- $\lim_{n\to\infty} a_n^p =$

We even have a version of the Squeeze Theorem for sequences: If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Using the Squeeze Theorem, what can you say about $\lim_{n\to\infty} a_n$ given $\lim_{n\to\infty} |a_n| = 0$?

Let's gather a few more pieces of terminology.

- A sequence $\{a_n\}$ is called *increasing* if $a_n < a_{n+1}$ for all $n \geq 1$. In other terms, that means

$$a_1 < a_2 < a_3 < \cdots$$

- Similarly, a sequence $\{a_n\}$ is called *decreasing* if $a_n > a_{n+1}$ for all $n \geq 1$.
- If a sequence is either increasing or decreasing, we say the sequence is *monotonic*.
- A sequence $\{a_n\}$ is *bounded above* if there is a number M such that $a_n \leq M$ for all $n \geq 1$.
- A sequence $\{a_n\}$ is *bounded below* if there is a number m such that $a_n \geq m$ for all $n \geq 1$.
- If a sequence is bounded above and bounded below, then we say $\{a_n\}$ is a *bounded sequence*.

Suppose that $\{a_n\}$ is an increasing sequence with all values lying between -2 and 1. Does the sequence converge or diverge? If it converges, can you make any claims about the limit of the sequence?

Cool Fact:

Try to draw a picture (by graphing some arbitrary monotonic sequences and their bounds; there are two cases) to convince yourself that this cool fact is true!

Play with these new (or old) creatures called sequences. Soon, you'll be an expert!

1. Assuming the pattern of the first few terms continues, find a formula for the n^{th} term of the sequence. Classify each as arithmetic, geometric, or neither.

   (a) $\{3, 8, 13, 18, \ldots\}$

   (b) $\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\right\}$

   (c) $\{-1, 1, -1, 1, \ldots\}$

   (d) $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right\}$

2. For each of the sequences above, determine whether the sequence converges or diverges. Why? If the sequence converges, what does the sequence converge to?

3. For which values of r does the sequence $\{a_n\}$ converge where $a_n = cr^n$ for $c \neq 0$?

4. Write the first four terms of the following sequences.

   (a) $\{a_n\}$ where $a_n = n(n+1)$

   (b) $\{b_n\}$ where $b_n = b_{n-1} + b_{n-2}$, $b_0 = 1$, $b_1 = 1$

   (c) $\{c_n\}$ that is arithmetic with common difference 2 and initial term 13

5. Suppose that $\{b_n\}$ is an decreasing sequence with all values lying above 4. Does the sequence converge or diverge? If it converges, can you make any claims about the limit of the sequence?

6. Suppose that $\{c_n\}$ is a monotonic sequence with all values lying between 0 and π. Does the sequence converge or diverge? If it converges, can you make any claims about the limit of the sequence?

7. Determine whether each sequence below (defined by the general term) converges or diverges. If it converges, find the limit.

   (a) $a_n = \dfrac{n^3}{n^3+1}$

   (b) $a_n = \dfrac{n^3}{n+1}$

   (c) $a_n = \dfrac{3^{n+2}}{5^n}$

8. ☕ Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leq 2$ and is decreasing. Determine if the sequence is convergent or divergent. If the sequence converges, find the limit.

9. ☕ A sequence is defined recursively by

$$a_1 = 1 \qquad a_{n+1} = 1 + \frac{1}{1 + a_n}$$

Find the first eight terms of the sequence $\{a_n\}$. What do you notice about the odd terms and the even terms? By considering the odd and even terms separately, show that $\{a_n\}$ is convergent and deduce that

$$\lim_{n\to\infty} a_n = \sqrt{2}.$$

This gives the continued fraction expansion

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \cdots}}$$