Goal: Use separable equations to model logistic growth.

In the last section, we used exponential growth to model population growth. That model assumed that the rate of growth increased linearly as the population grew. In some situations this is a reasonable model, but not always. Think about the amount of fish in a pond with no inlets or outlets. There's a limited amount of food for the fish, so their population can't increase exponentially forever. Eventually, the rate of growth will slow.

The *logistic model* gives us a model for population growth in a situation where the population has a *carrying capacity*, the largest population that the environment can support in the long run.

The growth rate for a population with carrying capacity M is given by the *logistic differential equation:*

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Some things to notice about this equation:

- When P is close to 0, the growth rate is close to kP, so population growth will be similar to ________________________.
  What does this mean in the context of initial population and how the population is changing?

- When P is close to M, the growth rate is ________________________.
  What does this mean in the context of initial population and how the population is changing?

- When P is larger than M, the growth rate is ________________________.
  What does this mean in the context of initial population and how the population is changing?

1. We want to solve the logistic differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$.

   (a) This differential equation is separable. Rewrite the equation with all the t terms on one side and all the P terms on the other.

   (b) Integrate both sides. On the P side, you will need to use partial fractions. Remember that P is the variable.

   (c) Solve for P in terms of M and some other constant—let's call it A.

   (d) Use the initial condition $P(0) = P_0$ to solve for A in terms of M and P_0.

2. Take the limit of your solution for $P(t)$ as t goes to infinity. Does this make sense in the context of the model?

3. Suppose there are 1,000 deer in a mountain range with a carrying capacity of 10,000 deer. If the population grows to 2,500 after one year, what will the population be after another three years?

   (a) Write out your solution to the logistic equation from the previous problem, plugging in 10,000 for M.

   (b) Use the initial population, $P(0) = P_0 = 1,000$ to find the constant A.

   (c) We still have an unknown in the equation $P(t)$, which is k. Use the second condition we have, $P(1) = 2,500$ to solve for k.

   (d) Use your equation for $P(t)$ with the value of k plugged in to find $P(4)$.

4. Here's a few more problems to practice your logistic model skills!

   (i) The population of the world was about 5.3 billion in 1990. The birth rate in 1990 was 35 million per year and the death rate was 15 million per year. Assume the carrying capacity of the earth is 100 billion.

   (a) Set up the logistic equation for the given values of M and P_0. Find k using the original differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ by assuming that $\frac{P}{M}$ is very close to zero.

   (b) Estimate the population in 2010.

   (c) Estimate the number of years until the population is 20 billion.

   (ii) Assume that the spread of a rumor in a town follows the logistic equation where P represents the percentage of people who have heard the rumor after t days.

   (a) What is the "carrying capacity" of P?

   (b) Suppose at $t = 0$, 20 % of people have heard the rumor. After one day, 50 % of people have heard the rumor. How long until 90 % of people have heard the rumor?