Goal: Use separable equations to model exponential growth and decay.

Many of our familiar formulas for modeling growth, decay, cooling, accumulated interest, etc., come from solving a related differential equation!

If $y(t)$ is the value of some quantity y at time t and the rate of change of y with respect to t is proportional to its size at $y(t)$ at any time, then $\frac{dy}{dt} = ky$ where k is some constant. If $k > 0$, we call this equation the *law of natural growth*. If $k < 0$, we call this equation the *law of natural decay*. Suppose $y(0) = y_0$ (this will represent some quantity at time zero). Let's solve this initial value problem:

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

There are a few named applications of this concept but we will only discuss two right now. Your text has more information on these and other applications.

Newton's Law of Cooling: Let $T(t)$ be the temperature of an object at time t. Let T_s be the temperature of the surroundings. The rate at which the object's temperature is changing is given by

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant.

1. A freshly brewed (but forgotten) cup of coffee has temperature 95°C in a 20°C room at 6:00 AM. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. At what time does this occur?

Radioactive Decay: Let $m(t)$ be the mass remaining from an initial mass m_0 of a radioactive substance after time t. The relative decay rate $-\frac{1}{m}\frac{dm}{dt}$ has been found to be constant. This information gives us the separable equation

$$\frac{dm}{dt} = km$$

where k is constant. (Quick check: In this situation, should k be positive or negative?)

2. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

   (a) Find the mass that remains after t years.

   (b) How much of the sample remains after 100 years?

   (c) After how long will only 1 mg remain?

3. Here's some more questions to help you become a modeling expert!

   (i) A curve passes through the point $(0,5)$ and has the property that the slope of the curve at every point P is twice the y-coordinate of P. What is the equation of the curve?

   (ii) A roast turkey is taken from an oven when its temperature has reached 165°F and is placed on a table in a room where the temperature is 70°F. After half and hour, the turkey has cooled to 120°F. What is the temperature after 45 minutes? When will the turkey have cooled to 90°F?

(iii) ☕ How long will it take an investment to double in value if the interest rate is 6% compounded continuously? What is the equivalent annual interest rate?

(iv) ☕ In order to determine the age of ancient organic material, scientists use a method called *radiocarbon dating*. All living things have approximately the same ratio of the stable carbon 12 (^{12}C) to the radioactive carbon 14 (^{14}C). Once something dies, it stops replenishing its carbon and the radioactive ^{14}C begins to decay. So we can use the current ratio to determine how long ago something stopped replenishing the decaying ^{14}C. The half life of ^{14}C is 5730 years. Say a fossil is found that has 35% ^{14}C compared to the living sample. How old is the fossil?