Goal: Solve separable equations analytically.

A *separable equation* is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y. That is,

$$\frac{dy}{dx} = g(x)f(y)$$

If $f(y)$ is never zero, we can equivalently write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

How do we solve these? That is, how do we find the general solution to a separable equation? Let's work through the general procedure together. Suppose that you are given a separable equation with an initial condition $y(x_0) = y_0$.

(A) Write your separable equation in factored form:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \quad h(y) \neq 0$$

(B) Move all of the y pieces to one side. (That is, multiply both sides by $h(y)$.)

(C) Integrate both sides with respect to x. (You should add both a $\int$ and a dx.)

(D) If $H'(y) = h(y)$ and $G'(x) = g(x)$, evaluate the integrals on both sides of the equation above. (You may want to look for an application of a familiar derivative rule.)

(E) Use your initial condition to solve for the constant of integration.

(F) If you can solve for y, you should! This isn't always possible and sometimes requires using the initial condition to make choices between different "branches" of the same function (think about $y^2 = x$ becomes $y = \pm\sqrt{x}$ which isn't a function).

Let's put this into practice!

1. Solve the differential equation.

   (a) $\dfrac{dy}{dx} = xy^2$

   (b) $\dfrac{dy}{dx} = xe^{-y}$

2. Find the solution of the differential equation that satisfies the given initial condition.

   (a) $\dfrac{dy}{dx} = \dfrac{x}{y}, \quad y(0) = -3$

   (b) $y' = \dfrac{\ln(x)}{xy}, \quad y(1) = 2$

3. Now that you're a separable equation pro, make sure to work some of these problems to keep improving your skills!

   (i) Find the general solution of the differential equation $(x^2+1)y' = xy$ [Hint: rewrite y' as dy/dx first.]

   (ii) Find the general solution of the differential equation $(y+\sin y)y' = x + x^3$

   (iii) Find the general solution of the differential equation $\frac{du}{dt} = 2 + 2u + t + tu$

(iv) Find the solution of the differential equation $\dfrac{dP}{dt} = \sqrt{Pt}$ that satisfies the initial condition $P(1) = 2$

(v) Find the solution of the differential equation $\dfrac{du}{dt} = \dfrac{2t + (\sec t)^2}{2u}$ that satisfies the initial condition $u(0) = -5$

(vi) Find an equation of the curve that passes through the point $(0, 1)$ and whose slope at (x, y) is xy.