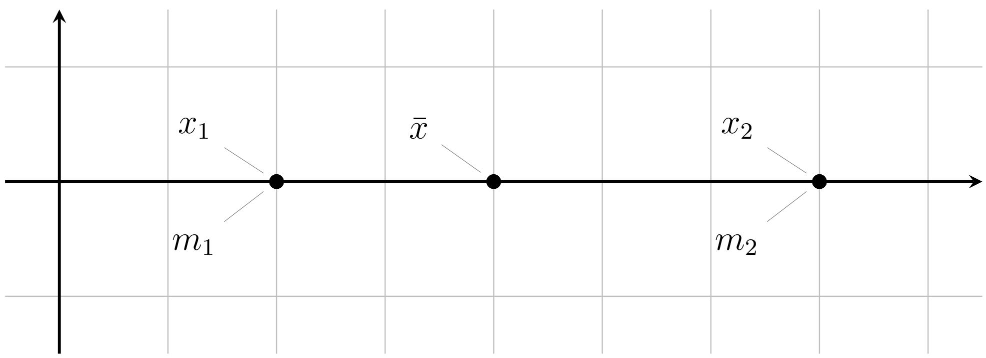
Goal: Compute moments and the center of mass for individual masses and laminas.

Let's start with *individual masses.* Perhaps you recall that to balance a rod with two different masses at each end, call them m_1 and m_2, we need to find the center of mass. Let's call the center of mass $\bar{x}$. If d_1 is the distance from m_1 to $\bar{x}$ and d_2 is the distance from m_2 to $\bar{x}$, then $\bar{x}$ has the property $d_1 m_1 = d_2 m_2$. Let's draw a picture of this scenario:

Now let's put this situation into our playing field—the Cartesian plane.



Suppose our rod now lies along the x axis (to the right of the origin). Call the point where m_1 lies x_1 and the point where m_2 lies x_2. Then we can rewrite the distances $d_1 = \bar{x} - x_1$ and $d_2 = x_2 - \bar{x}$. Since we must have $d_1 m_1 = d_2 m_2$, we can determine where the center of mass must lie:

$$\begin{aligned} d_1 m_1 &= d_2 m_2 \\ (\bar{x} - x_1) m_1 &= (x_2 - \bar{x}) m_2 \\ \bar{x} m_1 - x_1 m_1 &= x_2 m_2 - \bar{x} m_2 \\ \bar{x} &= \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \end{aligned}$$

Seeing the symmetry in this formula for the center of mass. I hope you are not too surprised that if there were 3 weights, then we would have

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

and so in general if we have n masses on a rod:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots m_nx_n}{m_1 + m_2 + m_3 + \cdots m_n} = \frac{\displaystyle\sum_{i=1}^{n} m_ix_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

Now you may ask, "What if our masses lie in a plane rather than on a line?" Well, do everything component-wise. That is, you compute the x component of the center of mass using the masses and the x coordinates of the masses and then do the same thing for y.

$$\bar{x} = \frac{\displaystyle\sum_{i=1}^{n} m_ix_i}{\displaystyle\sum_{i=1}^{n} m_i} \qquad \bar{y} = \frac{\displaystyle\sum_{i=1}^{n} m_iy_i}{\displaystyle\sum_{i=1}^{n} m_i}$$

The numerators above are called the moments of the system about the y and x axis respectively and are denoted by M_y and M_x. Notice M_y, the moment about the y axis is the obtained using the x coordinates, and M_x is obtained with the y coordinates.

What if we want to find the center of mass for something more realistic than floating isolated weights? In particular, we will look at flat regions of uniform density which we call *laminas*.

Like with the point masses, we compute the x and y components of the center of mass (known as the **centroid**) separately. We restrict ourselves to the case that the region we deal with can be described by a function of x or y. We'll stick to functions of x here. The equations look a little bizarre at first:

$$\bar{x} = \frac{1}{A}\int_a^b xf(x)\ dx \qquad \bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\ dx$$

Here A is the area of the region. Just as before, we have the moment about the x-axis, M_x, and the moment about the y-axis, M_y, which are below where ρ is the density of the plate.

$$M_x = \rho\int_a^b \frac{1}{2}[f(x)]^2\ dx, \qquad M_y = \rho\int_a^b xf(x)\ dx.$$

Something nice about how these definitions interact is that if m is the mass of the plate, then we have

$$m = \rho A = \rho\int_a^b f(x)\,dx, \qquad m\bar{x} = M_y, \qquad m\bar{y} = M_x.$$

Let's get rolling with some exercises.

1. Suppose there are masses $m_1 = 6, m_2 = 5$ and $m_3 = 10$ located at $(1, 5), (3, -2)$ and $(-2, -1)$ respectively. Find the moments M_x and M_y and the center of mass of the system.

2. Find the center of mass of the region bounded by $\sin(x)$ and the x axis on $[0, \pi]$.

3. Here are a few more problems to help you on your way to mastering moments and centers of mass.

   (i) Determine the center of mass for the region bounded by the curves $y = \sqrt{x}$ and $y = x^3$.

(ii) Find the center of mass of the region bounded by $y = 4 - x^2$ that is in the first quadrant.

(iii) Find the center of mass of the triangle with vertices $(0, 0)$, $(-4, 2)$, and $(0, 6)$.

(iv) Find the center of mass of the region bounded by $y = e^{2x}$ and $y = -\cos(\pi x)$ between $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

(v) ☕ Let $\mathscr{R}$ be the region that lies between the curves $y = x^m$ and $y = x^n$, $0 \leq x \leq 1$, where m and n are integers with $0 \leq n < m$.

(a) Sketch the region $\mathscr{R}$.

(b) Find the coordinates of the centroid of $\mathscr{R}$.

(c) Find values of m and n such that the centroid lies outside $\mathscr{R}$ or prove that no such pair m and n exist.