Goal: Given a solid described by rotating an area, compute its volume.

A General Procedure (Solids of Revolution)

(i) Draw a graph of the relevant functions/regions in the plane. Draw a vertical line and horizontal line within the region for reference. (This will look the same way as it would when you first learned integration and you used lines to indicate which direction you were integrating.

(ii) Indicate the axis of revolution and sketch how your solid might look. The better this picture is, the easier this problem will feel.

(iii) Add either a vertical or horizontal slice by following how your reference lines "travelled" when the area was rotated to create the solid. (It's worth the practice to chase both through to see what happens or which integral is easier to compute.) Think about which way we need to "thicken" the slice or rather, are we integrating dx or dy?

(iv) Identify what the area of this shape is. Did your slice create a disc, washer, or cylinder? What are the area formulas for each of these shapes?

(v) Set up the integral $\int_a^b A(x)dx$ or $\int_c^d A(y)\,dy$ to find the volume. Your picture will tell you the bounds.

Some Useful Area Formulas

(i) Disc: If we have a disc of radius r, the area is

$$A = \pi r^2.$$

(ii) Washer: If we have a washer of outer radius R and inner radius r, that's just a big disc with a smaller disc removed so the area is $\pi R^2 - \pi r^2$ or

$$A = \pi(R^2 - r^2).$$

(iii) Cylinder: If we have a cylinder of height h and radius r, we can compute the surface area by treating it as a rectangle of height h and length determined by the circumference of the circle of radius r which is $2\pi r$. So, the surface area of a cylinder is given by

$$A = 2\pi rh.$$

1. Let A be the area bounded by $f(x) = 1 - x^2$, the x-axis, and the y-axis. Find the volume of the solid of revolution formed by rotating A about the y-axis.

   (i) Draw A and a horizontal line for reference.

   (ii) Sketch the solid of revolution on the same axis.

   (iii) Sketch the slice created by your reference line including any relevant dimensions. Are we integrating dx or dy?

   (iv) Write a function for the area of this slice. Make sure the function is in terms of the variable you are integrating.

   (v) Set up the integral $\int_a^b A(x)dx$ or $\int_c^d A(y)\,dy$ to calculate the volume.

   (vi) Repeat steps (i)–(v) using a vertical reference line instead. Check that your integrals are equal either by hand or with your choice of internet resource.

2. Let A be the area bounded by $y = x^2 - 4$ and the x and y axes. Consider the solid formed by rotating A about the y-axis. Use both the disc/washer method and the cylindrical shell methods to compute the volume of this solid.

3. Let A be the area bounded by $f(x) = \dfrac{x^2}{3}$ and $g(x) = x$. Find the volume of the solid of revolution formed by rotating A about the x-axis.

4. Let A be the area bounded by $f(x) = \sin\left(x^2\right)$ and the x-axis. Find the volume of the solid of revolution formed by rotating A about the y-axis.

5. Find the volume of the solid formed by rotating the area bounded by $f(x) = (x-2)^3$, the x-axis, and $x = 3$ about $x = 1$.

6. Let A be the area bounded by $y = 1/x$, $y = 0$, $x = 1$ and $x = 2$. Find the volume of the solid of revolution formed by rotating A about the y-axis.

7. Here are some more solids whose volume you can compute for practice!

   (i) Find the volume of the solid formed by rotating the area bounded by $y = \frac{1}{2}x - 1$ and the x and y axes about $y = -3$.

   (ii) Let A be the area bounded by $y = 3 + 2x - x^2$ and $x + y = 3$. Find the volume of the solid of revolution formed by rotating A about the y-axis.

   (iii) Let A be the area bounded by $y = x^3$, $y = 0$, and $x = 1$. Find the volume of the solid of revolution formed by rotating A about $y = 1$.

(iv) ♨ Consider the area bounded by the functions $y = \sqrt{x}$ and $y = 2 - \sqrt{x}$ and the x-axis. Set up but do not evaluate integrals (using both the disc/washer method and the cylindrical shell method) which give the volumes of the solids created by revolving the area about each of the lines below. (This means your answer must include twelve integrals.)

$$y = 0, \quad x = 0, \quad y = 1, \quad x = 2, \quad x = 3, \quad y = -4.$$

(v) ☕ Consider the area bounded by the functions $y = \sqrt{x}$ and $y = 2 - \sqrt{x}$ and the y-axis. Set up but do not evaluate integrals (using both the disc/washer method and the cylindrical shell method) which give the volumes of the solids created by revolving the area about each of the lines below. (This means your answer must include twelve integrals.)

$$y = 0, \quad x = 0, \quad y = -2, \quad x = -2, \quad y = 4, \quad x = 4.$$