Goal: Given a description of a solid (base and cross-section), compute its volume.

We will develop three methods for computing volumes of solids depending on how they are described. First, we will be looking at solids that can be described by their bases and what "slices" (or cross-sections) look like. For example, if you chop a sphere into slices, you'd have circles.

Think about this: to find areas, we could integrate lengths. We sort of "thickened up" lengths (using dx or Δx) and integrated to obtain the areas. Now, we want to find volumes so we are going to "thicken up" an area and integrate.

Let's try an example and then outline the general procedure.

1. Consider a 3-dimensional solid S where the base of S is the region enclosed by the parabola $y = 1 - x^2$ and the x-axis and cross-sections perpendicular to the x-axis are squares.

   (a) Sketch the region of the base of S and include a cross-section of S.

   (b) Find the area $A(x)$ of the cross section of S at x. (This is what we call an arbitrary cross-section.)

   (c) By integrating $A(x)$ over an appropriate interval, compute the volume of the solid S.

A General Procedure (Slicing/Cross-sections)

(i) Draw a graph of the relevant functions in the plane and draw the relevant slice. The problem will tell you the direction. Then try to draw a picture of this region (with axes and slice) lying down in 3-D space. This takes some practice.

(ii) Now try to draw what a planar slice of the volume looks like.

(iii) Identify what the area of this shape is.

(iv) Set up the integral $\int_a^b A(x)dx$. The picture tells you your bounds.

2. Consider a 3-dimensional solid S whose base is the triangular region with vertices $(0,0)$, $(1,0)$, and $(0,2)$ and the cross-sections perpendicular to the y-axis are equilateral triangles.

(a) Sketch the region of the base of S and include a cross-section of S.

(b) Find a formula for the area of an equilateral triangle with side-length a.

(c) Using the formula from part (b), find the area $A(y)$ of the cross section of S at y.

(d) By integrating $A(y)$ over an appropriate interval, compute the volume of the solid S.

3. Consider a 3-dimensional solid S whose base is the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$ and the cross-section perpendicular to the x-axis are squares. Find the volume of S.

4. Find the volume of the solid S which is a right circular cone with height h and base radius r.

5. Here are some integrals to help you get a handle on the slicing process.

   (i) ☕ Find the volume of the solid S whose base is a circular disk with radius r and the parallel cross-sections perpendicular to the base are squares.

(ii) ☕ Compute the volume of each of the solids generated when the base is bounded by the circle $x^2 + y^2 = 4$ and the indicated cross-sections are taken perpendicular to the x-axis:

i. Cross sections are squares
ii. Cross sections are equilateral triangles
iii. Cross sections are semicircles
iv. Cross sections are isosceles right triangles (the 90-degree angle is at the vertex of the solid)