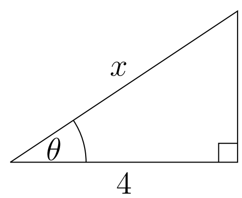
Goal: Evaluate integrals using substitutions involving trigonometric functions.

Let's jump right in with an example and build a general strategy later.

Example: Evaluate $\int \frac{\sqrt{x^2 - 16}}{x}\, dx$.

(a) First, let's draw a seemingly random picture. Label the length of the blank side.



(b) Compute $\sec(\theta)$ and solve for x.

(c) Your new expression for x is going to be our substitution. What is dx? (Remember the chain rule! Your expression should have a $d\theta$.)

(d) Let's return to our original integral and make these substitutions.

$$\int \frac{\sqrt{x^2 - 16}}{x}\, dx = \int \frac{\sqrt{\left(\qquad \right)^2 - 16}}{\left(\qquad \right)} \cdot \left(\qquad \qquad \right) d\theta$$

(e) With our new θ-integral, simplify the expression within the square root until you can utilize the formula $\big(\tan(\theta)\big)^2 = \big(\sec(\theta)\big)^2 - 1$.

(f) Now, you should have an integral that is a product of $\sec(\theta)$'s and $\tan(\theta)$'s which is an integral you can compute! Do that now.

(g) Now, you have a function in terms of θ but our question was asked in terms of x so we need to return to our original variable. You should start by computing $\tan(\theta)$ from our triangle and computing θ using an inverse trigonometric function (and remember that some of these have weird domains).

A General Procedure

1. Set up the right right triangle in preparation to use trigonometric functions. See the diagrams below for the three cases.
2. Solve for x by relating the side involving x with the side involving the constant term.
3. Compute dx.
4. Find an expression for the square root term in the same manner as step 2.
5. Now plug in your substitutions and solve the integral (hopefully).

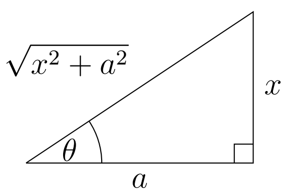


Figure 1: For terms involving $\sqrt{x^2+a^2}$

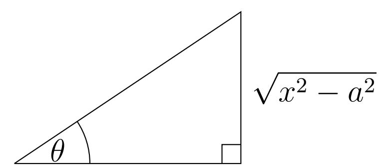


Figure 2: For terms involving $\sqrt{x^2-a^2}$

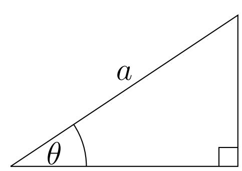


Figure 3: For terms involving $\sqrt{a^2-x^2}$

Fill in the missing side lengths of the triangles and complete Step 2 for each of the triangles.

1. Evaluate $\displaystyle\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\, dt$.

2. Evaluate $\int \frac{x^3}{\sqrt{x^2+1}}\, dx$.

3. Evaluate $\int \sqrt{x^2 + 4x + 3}\, dx$. Hint: You need to complete the square.

4. Here are some integrals to hone your new trigonometric substitution tool:

   (i) $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

   (ii) $\displaystyle\int_{-4/5}^{-2/5} \frac{\sqrt{25-x^2}}{x}\,dx$

   (iii) $\displaystyle\int \frac{\sqrt{y^2+9}}{y^4}\,dx$

(iv) $\displaystyle\int \frac{1}{x^2\sqrt{4-9x^2}}\ dx$

(v) ☕ $\displaystyle\int e^{2x}\sqrt{64-e^{2x}}dx$

(vi) ☕ $\displaystyle\int \frac{\sqrt{1-x}}{\sqrt{x}}dx$