Goal: Define an improper integral and be able to determine if a given improper integral converges or diverges directly from the definition. Also, if a given improper integral converges, be able to determine its value.

Recall that a definite integral represents an area under a curve between two values (as long as the curve is defined at all of those values) and is given by the following formula:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a),$$

where $F(x)$ is an antiderivative of $f(x)$.

We want to investigate what happens if $\int_a^t f(x)\,dx$ is defined for all values of t such that $t \geq a$. Well, let's allow t to approach ∞ using a limit. We define

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx.$$

So, if we know how to compute $\int_a^t f(x)\,dx$, all we have to do is add a limit! Graphically, this would represent the total area under a curve that is to the right of $x = a$. It is possible that this doesn't exist. (Can you create a few criteria that help you decide if this could exist or not?)

We can similarly define

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx.$$

But wait, there's more! How should we define the following integral?

$$\int_{-\infty}^\infty f(x)\,dx =$$

We call any integral with ∞ or $-\infty$ in the limits of integration (i.e., integrating over an infinite interval) an *improper integral* since we need to use limits to evaluate them.

We call an improper integral *convergent* if the limit exists and *divergent* otherwise.

1. Determine if $\int_1^{\infty} \frac{1}{x}\,dx$ converges or diverges.

2. Determine if $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges or diverges.

3. Determine if $\int_1^{\infty} \frac{1}{x^{10}}\,dx$ converges or diverges.

4. Determine if $\int_1^{\infty} \frac{1}{x^{1/2}}\,dx$ converges or diverges.

5. Determine if $\int_1^{\infty} \frac{1}{x^{1/3}}\,dx$ converges or diverges.

6. For what values of p does the integral $\int_1^{\infty} \frac{1}{x^p}\,dx$ converge? (We call this the p-test for improper integrals.)

There is a second type of improper integral: the integrand is discontinuous.

If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, we define

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx.$$

If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, we define

$$\int_a^b f(x)\,dx =$$

If $f(x)$ is continuous on $[a, c)$ and $(c, b]$ but discontinuous at c (where $a < c < b$), we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx =$$

7. Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\,dx$ and $\displaystyle\int_{-1}^1 \frac{1}{\sqrt[3]{x}}\,dx$.

8. Determine if $\displaystyle\int_{-1}^1 \frac{1}{x}\,dx$ converges or diverges.

9. Determine if $\int_3^\infty \frac{x^2}{x^4+4}\,dx$ converges or diverges. If the integral converges, determine its value.

10. Determine if $\int_0^\infty \frac{1}{(x-2)^2}\,dx$ converges or diverges. If the integral converges, determine its value.

11. Determine if $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$ converges or diverges. If the integral converges, determine its value.

12. Determine if $\int_{0}^{\infty} xe^{-x}\, dx$ converges or diverges. If the integral converges, determine its value.

13. Determine if $\int_0^8 \frac{1}{\sqrt[3]{8-x}}\,dx$ converges or diverges. If the integral converges, evaluate the integral.

14. Determine if $\int_0^1 \ln(x)\,dx$ converges or diverges.