Analysis Prelim January 2015

Sarah Arpin
University of Colorado Boulder
Mathematics Department
Sarah.Arpin@colorado.edu

Problem 1

If $g : [0, \infty) \to \mathbb{R}$ is a monotone non-increasing (thus measurable) function satisfying $\lim_{x\to\infty} g(x) = c > 0$, prove that there exists a rational-valued function $h : [0, \infty) \to \mathbb{Q}$ such that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f = g \cdot h$ is improperly Riemann integrable on $[0, \infty)$, but not Lebesgue integrable there.

Solution:
The function $g(x) := \frac{1}{x} + 1$ satisfies $\lim_{x\to\infty} g(x) = 1 > 0$, and $g(x)$ is nonincreasing.
Define the function $h : [0, \infty) \to \mathbb{Q}$:

$$h(x) := \begin{cases} 0 & \text{for } x \in [0, 1) \\ \frac{(-1)^n}{n} & \text{for } x \in [n, n+1), \text{ for } n = 1, 2, 3, ... \end{cases}$$

Then, the function $f = g \cdot h$ is given:

$$f(x) := \begin{cases} 0 & \text{for } x \in [0, 1) \\ \frac{(-1)^n}{n}\left(\frac{1}{x} + 1\right) & \text{for } x \in [n, n+1), \text{ for } n = 1, 2, 3, ... \end{cases}$$

Looking at the improper Riemann integral:

$$\begin{aligned} \int_0^\infty f(x)dx &= \sum_{n=1}^\infty \left(\int_n^{n+1} \frac{(-1)^n}{n}\left(\frac{1}{x} + 1\right) dx \right) \\ &\le \sum_{n=1}^\infty \left(\int_n^{n+1} \frac{(-1)^n}{n}\left(\frac{1}{n} + 1\right) dx \right) \\ &= \sum_{n=1}^\infty \frac{(-1)^n}{n}\left(\frac{1}{n} + 1\right) \\ &= \sum_{n=1}^\infty \frac{(-1)^n}{n^2} + \sum_{n=1}^\infty \frac{(-1)^n}{n} \\ &< \infty, \text{ bc both series are convergent by the alternating series test.} \end{aligned}$$

So f is Riemann integrable on $[0, \infty)$.
To be Lebesgue integrable on $[0, \infty)$, we need to show

$$\int_0^\infty |f(x)|dx < \infty$$

Which does not hold:

$$\begin{aligned} \int_0^\infty |f(x)|dx &= \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{n}\left(\frac{1}{x} + 1\right) dx \\ &\ge \sum_{n=1}^\infty \int_n^{n+1} \frac{1}{n}\left(\frac{1}{n+1} + 1\right) dx \\ &= \sum_{n=1}^\infty \frac{1}{n(n+1)} + \sum_{n=1}^\infty \frac{1}{n} \\ &\to \infty, \text{ because the harmonic series diverges.} \end{aligned}$$

So f is not Lebesgue integrable on $[0, \infty)$.

$\square$

Problem 2

Assume that $f : [1, 2] \to \mathbb{R}$ is absolutely continuous, with $f(2) = 0$. Prove that

$$\left| \int_1^2 f'(x) \log(x) dx \right| \leq \int_1^2 |f(x)| dx$$

Solution:
Consider the left-hand side, using integration by parts:

$$\begin{aligned}
\left| \int_1^2 f'(x) \log(x) dx \right| &= \left| \log(x) f(x) \Big|_1^2 - \int_1^2 \frac{f(x)}{x} dx \right| \\
&= \left| \log(2) f(2) - \log(1) f(1) - \int_1^2 \frac{f(x)}{x} dx \right| \\
&= \left| \int_1^2 \frac{f(x)}{x} dx \right| \\
&\leq \int_1^2 \left| \frac{f(x)}{x} \right| dx \\
&\text{And on } [1, 2], \left| \frac{f(x)}{x} \right| \leq |f(x)|, \text{ so:} \\
&\leq \int_1^2 |f(x)| dx
\end{aligned}$$

$\square$

Problem 3

Let $f : [a, b] \to \mathbb{R}$ be a C^1 function.
For $\epsilon > 0$, let $C_\epsilon := \{x \in (a, b) : |f'(x)| < \epsilon\}$, and let $A := \{f(x) : x \in (a, b), f'(x) = 0\}$.

(i) Prove that C_ϵ is open and that $m(f(C_\epsilon)) < \epsilon \cdot (b - a)$.

(ii) Prove that A has Lebesgue measure zero.

Solution:

(i) Since f is C^1, we know that f' is a continuous function. Since C_ϵ is the pre-image of an open set in $\mathbb{R}$ under the continuous function f', C_ϵ is open.
Since C_ϵ is open, it can be written as a disjoint union of open intervals:

$$C_\epsilon = \bigcup_{k=1}^{\infty} (a_k, b_k)$$

For any k and any $x \in (a_k, b_k)$, we have $|f'(x)| < \epsilon$. This gives us a bound on the value of $f((a_k, b_k))$.
For $x \in (a_k, b_k)$:

$$|f(a_k)| - (b_k - a_k)\epsilon < |f(x)| < |f(a_k)| + (b_k - a_k)\epsilon$$

For each k, $f((a_k, b_k)) \subseteq (|f(a_k)| - (b_k - a_k)\epsilon, |f(a_k)| + (b_k - a_k)\epsilon)$.
We can do better than that though!

Let $m_k := \min\{f(x) : x \in [a_k, b_k]\}$. Then the maximum value of $f(x)$ on (a_k, b_k) is less than $m_k + (b_k - a_k)\epsilon$, so we can further refine this inclusion:

$$f((a_k, b_k)) \subseteq (m_k, m_k + (b_k - a_k)\epsilon)$$

Looking at the integral definition of measure:

$$\begin{aligned}
m(f(C_\epsilon)) &= \int_{f(C_\epsilon)} 1dm \\
&\leq \sum_{k=1}^{\infty} \int_{f((a_k,b_k))} 1dm \\
&\leq \sum_{k=1}^{\infty} \int_{m_k}^{m_k+(b_k-a_k)\epsilon} 1dm \\
&= \epsilon \sum_{k=1} (b_k - a_k) \\
&\leq \epsilon(b - a)
\end{aligned}$$

(ii) Sard's Theorem?
We revisit the C_ϵ, but for our purposes it is easier to define $C_{1/n}$.
Note that $C_1 \supseteq C_{1/2} \supseteq C_{1/3} \supseteq \cdots$. So we have a descending chain of open sets.
By the property of measure, we have

$$m(\lim_{n\to\infty} C_{1/n}) = \lim_{n\to\infty} m(C_{1/n})$$

Note that $A = \lim_{n\to\infty} f(C_{1/n})$

$$\begin{aligned}
m(A) &= m\left(\lim_{n\to\infty} f(C_{1/n})\right) \\
&= \lim_{n\to\infty} m(f(C_{1/n})) \\
&\leq \lim_{n\to\infty} \epsilon \cdot (b - a) \\
&= 0
\end{aligned}$$

□

Problem 4

Let $(X, \mathcal{B}, \mu)$ be a measure space, and suppose that $p, q, r \in (1, \infty)$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$$

If $f \in L^p(X, \mu), g \in L^q(X, \mu)$, and $h \in L^r(X, \mu)$, prove that $f \cdot g \cdot h \in L^1(X, \mu)$ and that

$$\|f \cdot g \cdot h\|_1 \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r$$

Solution:

To prove this theorem, we will use the fact that:

$$a^{1/p} b^{1/q} c^{1/r} \leq \frac{1}{p}a + \frac{1}{q}b + \frac{1}{r}c$$

for p, q, r satisfying the above identity and for any nonnegative real numbers a, b, c.
Assuming this, we can use this inequality with the substitutions:

$$a = \left|\frac{f(x)}{\|f\|_p}\right|^p, \; b = \left|\frac{g(x)}{\|g\|_q}\right|^q, \text{ and } c = \left|\frac{h(x)}{\|h\|_r}\right|^r$$

Plugging these in:

$$\left|\frac{f(x)}{\|f\|_p}\right| \cdot \left|\frac{g(x)}{\|g\|_q}\right| \cdot \left|\frac{h(x)}{\|h\|_r}\right| \leq \frac{1}{p}\left|\frac{f(x)}{\|f\|_p}\right|^p + \frac{1}{q}\left|\frac{g(x)}{\|g\|_q}\right|^q + \frac{1}{r}\left|\frac{h(x)}{\|h\|_r}\right|^r$$

Integrating both sides over X:

$$\frac{1}{\|f\|_p \|g\|_q \|h\|_r}\int_X |f(x)g(x)h(x)|dx \leq \frac{1}{p}\frac{\|f\|_p^p}{\|f\|_p^p} + \frac{1}{q}\frac{\|g\|_q^q}{\|g\|_q^q} + \frac{1}{r}\frac{\|h\|_r^r}{\|h\|_r^r}$$

$$\frac{1}{\|f\|_p \|g\|_q \|h\|_r}\int_X |f(x)g(x)h(x)|dx \leq 1$$

$$\int_X |f(x)g(x)h(x)|dx \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r$$

$$\|f \cdot g \cdot h\|_1 \leq \|f\|_p \cdot \|g\|_q \cdot \|h\|_r$$

As for the inequality that this result relies on, note that it follows from Jensen's inequality. For any $\lambda_1, \lambda_2, ..., \lambda_n \in (0,1)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ and any concave function φ:

$$\sum_{k=1}^{n} \lambda_k \varphi(x_k) \leq \varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right) \text{ for all } x_1, x_2, \ldots, x_n \text{ in the domain of } \varphi$$

This can be proven by induction:
Base Case: Suppose $\lambda_1 + \lambda_2 = 1$. By the definition of concave function we have:

$$\lambda_1\varphi(x_1) + \lambda_2\varphi(x_2) \leq \varphi(\lambda_1 x_1 + \lambda_2 x_2)$$

Inductive Step: Suppose the result holds for $\lambda_1, ..., \lambda_n$ such that $\lambda_1 + \cdots + \lambda_n = 1$. Show that it holds for $\lambda_1, ..., \lambda_n, \lambda_{n+1}$ such that $\lambda_1 + \cdots + \lambda_n + \lambda_{n+1} = 1$.

$$\varphi(\lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1}x_{n+1}) = \varphi\left(\lambda_1 x_1 + (1-\lambda_1)\sum_{k=2}^{n+1}\frac{\lambda_k x_k}{(1-\lambda_1)}\right)$$

Now we are looking at the base case, so we have:

$$\geq \lambda_1\varphi(x_1) + (1-\lambda_1)\varphi\left(\sum_{k=2}^{n+1}\frac{\lambda_k x_k}{(1-\lambda_1)}\right)$$

Note that we are now in the case of the inductive hypothesis for the φ expression on the r

$$\lambda_1 + \cdots + \lambda_{n+1} = 1 \Rightarrow \sum_{k=2}^{n+1}\frac{\lambda_k}{1-\lambda_1} = 1$$

So we can apply this inductive hypothesis to get:

$$\varphi(\lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1}x_{n+1}) \geq \lambda_1\varphi(x_1) + (1-\lambda_1)\sum_{k=2}^{n+1}\frac{\lambda_k\varphi(x_k)}{1-\lambda_1}$$

$$\varphi(\lambda_1 x_1 + \cdots + \lambda_n x_n + \lambda_{n+1}x_{n+1}) \geq \sum_{k=1}\lambda_k\varphi(x_k)$$

Which is the desired result.
Noting that log is a convex function and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, we can apply this result to our particular case:

$$\frac{1}{p}\log(a) + \frac{1}{q}\log(b) + \frac{1}{r}\log(c) \leq \log\left(\frac{1}{p}a + \frac{1}{q}b + \frac{1}{r}c\right)$$

Using the properties of logs:

$$\log(a^{1/p}b^{1/q}c^{1/r}) \leq \log\left(\frac{1}{p}a + \frac{1}{q}b + \frac{1}{r}c\right)$$

Since the exponential function is one-to-one and strictly increasing, we can apply it to both sides of the inequality and it remains preserved, yielding the desired result:

$$a^{1/p}b^{1/q}c^{1/r} \leq \frac{1}{p}a + \frac{1}{q}b + \frac{1}{r}c$$

And this holds for any a, b, c in the domain of log, so any positive a, b, c work. This justifies the above work, and yields the desired result.

$\square$

Way Easier Solution to Problem 4:
First, by Hölder's Inequality:

$$\|fgh\|_1 \leq \|f\|_p \|gh\|_{p'} \text{ where } \frac{1}{p} + \frac{1}{p'} = 1 \text{ for some} p' > 1 \tag{1}$$

Now we will look at $\|gh\|_{p'}$. First, set $\alpha = \frac{q}{p'}$ and $\beta = \frac{r}{p'}$. Then:

$$\frac{1}{\alpha} + \frac{1}{\beta} = p'\left(\frac{1}{q} + \frac{1}{r}\right) = p'(1 - \frac{1}{p}) = \frac{p'}{p'} = 1$$

This allows us to use Hölder's Inequality again.

$$\begin{aligned}
\|gh\|_{p'} &= \left(\int_{\mathbb{R}} |g|^{p'}|h|^{p'}\right)^{1/p'} \\
&\text{By Hölder's Inequality:} \\
&\leq \left(\left\||g|^{p'}\right\|_\alpha \left\||h|^{p'}\right\|_\beta\right)^{1/p'} \\
&= \left[\left(\int_{\mathbb{R}} |g|^q\right)^{p'/q}\left(\int_{\mathbb{R}} |h|^r\right)^{p'/r}\right]^{1/p'} \\
&= \left(\int_{\mathbb{R}} |g|^q\right)^{1/q}\left(\int_{\mathbb{R}} |h|^r\right)^{1/r} \\
&= \|g\|_q \cdot \|h\|_r
\end{aligned} \tag{2}$$

Combining (1) and (2):

$$\|fgh\|_1 \leq \|f\|_p \|g\|_q \|h\|_r$$

$\square$

Problem 5

Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and suppose that $f : X \to [0, \infty)$ is a nonnegative integrable function. Prove that the function $\psi : [0, \infty) \to [0, \infty]$ defined by $\psi(t) = \mu(\{x \in X : f(x) \geq t\})$ is Lebesgue measurable and that

$$\int_X f d\mu = \int_0^\infty \psi(t)dt$$

Hint: you may find Tonelli's Theorem useful.

Solution:
Begin by looking at the integral on the right side:

$$\int_0^\infty \psi(t)dt = \int_0^\infty \mu(\{x \in X : f(x) \geq t\})dt$$

By definition of measure:

$$= \int_0^\infty \left(\int_{f \geq t} 1dx \right) dt$$

We can switch the order of integration by Tonelli's Theorem (1 is clearly nonnegative)

Consider the effect this has on the bounds of the integrals:

$0 \leq t < \infty$ and $f(x) \geq t \Rightarrow t \leq f(x)$ and $0 \leq f(x) < \infty$

The bound on $f(x)$ is equivalent to $x \in X$, so:

$$= \int_X \int_0^{f(x)} 1dtdx$$

$$= \int_X f(x)dx$$

Which yields the desired result.

□

Problem 6

If $\{f_1, f_2, ...\}$ is a complete orthonormal set in the Hilbert space $L^2([0,1])$, where $[0,1]$ is equipped with the Lebesgue measure, and B is an arbitrary measurable subset of positive measure in $[0,1]$, use Parseval's identity applied to the characteristic function for B to prove that:

$$1 \leq \int_B \sum_{i=1}^\infty |f_i(x)|^2 dx$$

Solution:
By Parseval's Identity:

$$\begin{aligned}
\mu(B) = \|\chi_B\|^2_{L^2[0,1]} &= \sum_{i=1}^\infty |\langle \chi_B, f_i(x) \rangle|^2 \text{ (By Parseval's Identity)} \\
&= \sum_{i=1}^\infty \left(\int_0^1 \chi_B f_i \right)^2 \\
&= \sum_{i=1}^\infty \left(\int_0^1 \chi_B \cdot (\chi_B f_i) \right)^2 \\
&\leq \sum_{i=1}^\infty \left(\|\chi_b\|_{L^2[0,1]} \|\chi_B f_i\|_{L^2[0,1]} \right)^2 \\
&= \sum_{i=1}^\infty \|\chi_b\|^2_{L^2[0,1]} \|\chi_B f_i\|^2_{L^2[0,1]} \\
&= \sum_{i=1}^\infty \mu(B) \|\chi_B f_i\|^2_{L^2[0,1]}
\end{aligned}$$

So this shows:

$$\mu(B) \leq \sum_{i=1}^\infty \mu(B) \|\chi_B f_i\|^2_{L^2[0,1]}$$

Moving forward from here, we can divide by $\mu(B)$:

$$\mu(B) \leq \sum_{i=1}^{\infty} \mu(B) \left\| \chi_B f_i \right\|^2_{L^2[0,1]}$$

Dividing by $\mu(B)$:

$$1 \leq \sum_{i=1}^{\infty} \left\| \chi_B f_i \right\|^2_{L^2[0,1]}$$

$$= \sum_{i=1}^{\infty} \int_B |f_i|^2$$

By Tonelli's theorem, the integrand is positive so we can switch the sum and the integral:

$$= \int_B \sum_{i=1}^{\infty} |f_i|^2$$

□