Analysis Prelim August 2015

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Problem 1

Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx$$

exists and evaluate the limit. Does the limit always exist if f is only assume to be Lebesgue integrable?

Solution:
Note that if $x \in [0, 1]$, then $x^n \in [0, 1]$ for any $n \in \mathbb{N}$.
By the Extreme Value Theorem, since f is continuous on the closed interval $[0, 1]$, it must achieve a maximum value on that interval: $\sup\limits_{x\in[0,1]} |f(x)| = \|f\|_\infty < \infty$.
For any n:

$$\int_0^1 |f(x^n)|dx \leq \int_0^1 \|f\|_\infty \, dx = \|f\|_\infty$$

Since the righthand side is finite and independent of n, taking the limit as $n \to \infty$ we still get a finite value for the integral.
To find the value, we need to approximate f by polynomials, so first suppose $p(x)$ is a polynomial:

$$p(x) = \sum_{i=0}^{m} a_i x^k \text{ where } a_i \in \mathbb{R}$$

For any given n:

$$\int_0^1 p(x^n)dx = \int_0^1 \sum_{i=0}^{m} a_i x^{in} dx$$

By linearity of the integral:

$$= \sum_{i=0}^{m} a_i \int_0^1 x^{in} dx$$

$$= \sum_{i=0}^{m} a_i \left(\frac{x^{in+1}}{in+1}\Big|_0^1 \right)$$

$$= \sum_{i=0}^{m} \frac{a_i}{in+1}$$

Taking the limit as $n \to \infty$:

$$\lim_{n\to\infty} \int_0^1 p(x^n)dx = \lim_{n\to\infty} \sum_{i=0}^{m} \frac{a_i}{in+1} = 0$$

By the Stone-Weierstrass theorem, since $f \in C([0, 1])$, we can find a sequence of polynomials with coefficients in $\mathbb{R}$ such that $\{p_i\} \to f$ uniformly on $[0, 1]$.
For all n, if $x \in [0, 1]$, then $x^n \in [0, 1]$, so the same sequence of polynomials can be used to approximate

$f(x^n)$.
Fix $\epsilon > 0$:

$$\begin{aligned}\left|\int_0^1 f(x^n)dx - \int_0^1 p_i(x^n)dx\right| &\le \int_0^1 |f(x^n) - p_i(x^n)|dx \\ &\le \int_0^1 \|f - p_i\|_\infty \, dx \\ &= \|f - p_i\|_\infty\end{aligned}$$

As $i \to \infty$, $\|f - p_i\|_\infty \to 0$, so there exists $N \in \mathbb{N}$ such that if $i \ge N$, then $\|f - p_i\|_\infty < \epsilon$.
Taking $i \ge N$:

$$\left|\int_0^1 f(x^n)dx - \int_0^1 p_i(x^n)dx\right| \le \|f - p_i\|_\infty < \epsilon$$

This shows that the value of the integral of $f(x^n)$ is equal to the limit of the integral of the approximating polynomials:

$$\int_0^1 f(x^n)dx = \lim_{i\to\infty} \int_0^1 p_i(x^n)dx = 0$$

If f is Lebesgue integrable, then $f \in L^1([0,1])$. The continuous functions $C([0,1])$ are dense in L^1, so we can approximate this integral using a sequence of $C([0,1])$ functions to approximate the integrand.
If $f \in L^1([0,1])$ is chosen arbitrarily, we can find a sequence $\{g_i\} \subset C([0,1])$ such that $g_i \to f$ with respect to the L^1 norm on $[0,1]$.

$$\int_0^1 |f(x) - g_i(x)|dx \to 0 \text{ as } i \to \infty$$

Likeweise, $x^n \in [0,1]$ for any n, so we can use this sequence to approximate $f(x^n)$ for any n.
As we have shown, for each of the g_i functions

$$\lim_{n\to\infty} \int_0^1 g_i(x^n)dx$$

exists and is finite.
Now, we will show that the integral can be approximated for $f \in L^1([0,1])$. Fixing $\epsilon > 0$:

$$\begin{aligned}\left|\int_0^1 f(x^n)dx - \int_0^1 g_i(x^n)dx\right| &\le \int_0^1 |f(x^n) - g_i(x^n)|dx \\ &= \|f(x^n) - g_i(x^n)\|_{L^1}\end{aligned}$$

Since $\|f(x^n) - g_i(x^n)\|_{L^1} \to 0$ as $i \to \infty$, we can find $N \in \mathbb{N}$ such that for all $i \ge N$:

$$\|f(x^n) - g_i(x^n)\|_{L^1} < \epsilon$$

which yields the desired approximation.

□

Problem 2

Assume that a Lebesgue measurable set E is contained in the interval $[a, b]$ for some $0 < a < b < \infty$. Let $\delta > 1$. If the sets E and δE (the elements of E multiplied by δ) are disjoint, prove that the measure of E is at most $\frac{b}{2}\log(b\delta/a)$.

Solution:
We will justify the following (in)equalities, which yields the desired result:

$$\frac{2}{b}\mu(E) \underbrace{=}_{1} \frac{1}{b}\mu(E)+\frac{1}{\delta b}\mu(\delta E) \underbrace{\leq}_{2} \int_E \frac{1}{x}dx+\int_{\delta E}\frac{1}{x}dx \underbrace{=}_{3} \int_{E\cup\delta E}\frac{1}{x}dx \underbrace{\leq}_{4} \int_a^{\delta b}\frac{1}{x}dx = \log(\delta b)-\log(a) = \log(\delta b/a)$$

1: For the first inequality, we need to show that $\mu(\delta E) = \delta\mu(E)$. This comes from the definition of measure. If we suppose that E is an open set, then it can be written as a union of disjoint open intervals $\cup_k(a_k, b_k)$ and its Lebesgue measure is given:

$$\mu(E) = \sum_k (b_k - a_k)$$

Since the set δE contains the elements of E multiplied by δ, δE can be written as the union $\cup_k(\delta a_k, \delta b_k)$, so its Lebesgue measure is given:

$$\mu(\delta E) = \sum_k(\delta b_k - \delta a_k) = \delta\sum_k(b_k - a_k) = \delta\mu(E)$$

Likewise, if E is closed, we can pull out the δ from the definition of Lebesgue measure to conclude that $\delta\mu(E) = \mu(\delta E)$.

2: Since the function $1/x$ is strictly decreasing on (0∞), it is strictly decreasing on $[a, b]$.
On $E \subseteq [a, b]$, the function $1/x$ will always be greater than or equal to $1/b$. Thus:

$$\int_E \frac{1}{x}dx \geq \int_E \frac{1}{b}dx = \frac{1}{b}\mu(E)$$

On $\delta E \subseteq [\delta a, \delta b]$, the function $1/x$ will always be greater than or equal to $1/\delta b$. Thus:

$$\int_{\delta E} \frac{1}{x}dx \geq \int_{\delta E} \frac{1}{\delta b}dx = \frac{1}{\delta b}\mu(\delta E)$$

3: Since E and δE are disjoint, we can use the additive property of the integral for disjoint domains.

4: Since $E \cup \delta E \subseteq [a, \delta b]$ and the integrand function is always positive on $[a, \delta b]$, we can again use the additivity of the integral to get this part of the inequality.

□

Problem 3

(i) Find a sequence of continuous functions on $[0, 1]$ converging pointwise but not uniformly.

(ii) Prove that the space $C([0, 1])$ of continuous functions on $[0, 1]$ is not complete in the L^1 metric $d(f, g) = \int_0^1 |f(x) - g(x)|dx$.

Solution:

(i) Define f_n via:

$$f_n(x) = \begin{cases} 0 & \text{for } x \in [0, 1 - \frac{1}{n}) \\ n(x-1)+1 & \text{for } x \in [1 - \frac{1}{n}, 1] \end{cases}$$

f_n converges pointwise to the function

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{for } x = 1 \end{cases}$$

$f_n(1) = 1 = f(1)$ for all n, and if $x \in [0,1)$, then there exists N such that $x < 1 - \frac{1}{N}$, and for all $n \geq N$ we will have $f_n(x) = 0$.
Since each function f_n is continuous, if $\{f_n\}$ converged uniformly it would converge to a continuous function. Since it converges to a function which is not continuous, this convergence is not uniform.

(ii) We need to find a sequence of functions in $C([0,1])$ that converges to something in $L^1([0,1]) \setminus C([0,1])$, with respect to the L^1 metric. Define f_n:

$$f_n(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 - n(x - \frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n} \\ 0 & \text{if } \frac{1}{2} + \frac{1}{n} \leq x \leq 1 \end{cases}$$

These functions are individually continuous, but they converge in L^1 sense to $f(x)$ defined:

$$f(x) := \begin{cases} 1 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

This function f is in $L^1([0,1]) \setminus C([0,1])$.
To see the L^1-convergence:

$$\begin{aligned} \|f - f_n\|_{L^1} &= \int_0^1 |f(x) - f_n(x)| dx \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} |1 - 1 + n(x - \frac{1}{2})| dx \\ &= n \int_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} (x - \frac{1}{2}) dx \\ &= n \left(\frac{x^2}{2} - \frac{1}{2} x \Big|_{\frac{1}{2}}^{\frac{1}{2}+\frac{1}{n}} \right) \\ &= \frac{1}{2n} \end{aligned}$$

Taking the limit as $n \to \infty$, $\|f - f_n\|_{L^1} \to 0$. This shows L^1 convergence, but clearly $f \notin C([0,1])$.

□

Problem 4

Let $\{\varphi_n\}$ be a sequence of continuous real-valued functions defined on a compact metric space X. For each $x \in X$, suppose that the sequence of values $\{\varphi_n(x)\}$ is non-decreasing and bounded above. Define

$$\varphi(x) = \lim_{n \to \infty} \varphi_n(x)$$

If φ is continuous, prove that the sequence $\{\varphi_n\}$ converges uniformly to φ.

Solution:
Fix $\epsilon > 0$.
Define the sequence of functions g_n:

$$g_n(x) := \varphi(x) - \varphi_n(x)$$

Since $\{\varphi_n(x)\}$ is a nondecreasing sequence with limit $\varphi(x)$, the sequence of g_n functions is nonincreasing as $n \to \infty$.
Define the open sets E_n:

$$E_n(x) := \{x \in X : g_n(x) < \epsilon\}$$

These sets are open, because g_n is continuous and the E_n are the preimages of open sets under g_n.
Also, $X = \cup_n E_n$, because every $x \in X$ is eventually in some E_n, since $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$.
Thus, $\{E_n\}_{n=1}^{\infty}$ forms an open cover of X. Since X is compact, there exists some finite subcover: $\{E_n\}_{n=1}^{N}$, for some $N \in \mathbb{N}$.
Thus, for all $x \in X$:

$$\varphi(x) - \varphi_N(x) = g_N(x) < \epsilon$$

Which shows that $\varphi_n \to \varphi$ uniformly.

□

Problem 5

Let M be a bounded subset of $C([a, b])$, the set of continuous functions on $[a, b]$ equipped with the sup norm. Set

$$A = \left\{ F : [a, b] \to \mathbb{R} : F(x) = \int_a^x f(t)dt \text{ for some } f \in M \right\}$$

Show that the closure of A is a compact subset of $C([a, b])$.

Solution:
By Arzela-Ascoli, to show that $\overline{A}$ is compact, we need to show it is closed, bounded and equicontinuous.
A note on M: Since M is bounded, there exists some real number m such that $\|f\|_\infty \le m$ for all $f \in M$.
By definition, $\overline{A}$ is closed.
To show that $\overline{A}$ is bounded, consider an arbitrary $F \in A$.

$$\begin{aligned} |F(x)| &= \left| \int_a^x f(t)dt \right| \\ &\le \int_a^x |f(t)|dt \\ &\le \int_a^x m dt \\ &\le m|b-a| \end{aligned}$$

This shows that F is bounded by $m|b - a|$.
If $F \in \overline{A}$ but not in A, then there is a sequence $\{F_n\} \subseteq A$ which converges to F.
For any given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \ge N$ then $|F(x) - F_n(x)| < \epsilon$

$$\begin{aligned} |F(x)| &= |F(x) - F_n(x) + F_n(x)| \\ &\le |F(x) - F_n(x)| + |F_n(x)| \\ &\le \epsilon + m|b-a| \end{aligned}$$

Since ϵ can be chosen arbitrarily small, F is bounded by $m|b - a|$ as well.
To show that $\overline{A}$ is equicontinuous, we need to show that for any $x \in [a, b]$ and any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon \text{ for all } f \in \overline{A}$$

Fixing $x \in [a,b]$ and $\epsilon > 0$, consider $F \in \overline{A}$. If $F \in A$, then:

$$
\begin{aligned}
|F(x) - F(y)| &= \left| \int_y^x f(x)dx \right| \\
&\text{wlog, assume } x > y \text{ (otherwise, switch them).} \\
&\leq \int_y^x |f(x)|dx \\
&\leq \int_y^x m dx \\
&= |x - y|m
\end{aligned}
$$

So if we pick $\delta = \epsilon/m$, if $|x - y| < \delta$ then $|F(x) - F(y)| < \epsilon$ as desired.
If F is in the closure of A instead of the interior, then F is a limit point of some sequence of F_n in A, so:

$$|F(x) - F(y)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(y)| + |F_n(y) - F(y)|$$

The first and last terms on the right side of the above inequality can be made arbitrarily small, and we showed above that the middle term is less than ϵ for an appropriate choice of δ (for $\delta = \epsilon/(3m)$), so we have that the equicontinuity holds across all of $\overline{A}$. This *delta* will also work for F's in A, so we have equicontinuity. This shows that $\overline{A}$ is closed, bounded, and equicontinuous, so by Arzela-Ascoli $\overline{A}$ is a compact subset of $C([a,b])$.

□

Problem 6

Let f be a Lebesgue measurable real-valued function on the interval $(0,1)$. For $n = 1, 2, 3, ...$ assume that the integrals

$$\int_0^1 x(f(x))^n dx$$

exist and have the same nonzero value. Prove that $f(x) = 1$ on a set of positive measure and is otherwise almost everywhere zero.
Hint: First show that f is essentially bounded.

Solution:
First, we will show that $f(x) \leq 1$ a.e. on $[0,1]$.
If $f(x) > 1$ on a subset $A \subseteq [0,1]$, then $f(x)^{2n} \to \infty$ as $n \to \infty$ on A. If $m(A) > 0$, this would contradict the fact that we assume $\int_0^1 x(f(x))^n dx$ takes the same finite value for all values of n:

$$
\begin{aligned}
\int_0^1 (f(x))^{2n} x &\geq \int_A (f(x))^{2n} x dx \\
&\to \infty \text{ as } n \to \infty
\end{aligned}
$$

So $m(A) = 0$, and we have $\int_A x(f(x))^{2n} x dx = 0$ for all n. This shows $f(x) \leq 1$ almost everywhere on $[0,1]$.
Now, look at the two integrals when $n = 2$ and when $n = 3$. They should have the same finite value, so when we subtract them we will get 0:

$$
\begin{aligned}
\int_0^1 x(f(x))^2 dx - \int_0^1 x(f(x))^3 dx &= 0 \\
\int_0^1 x(f(x))^2 (1 - f(x)) dx &= 0
\end{aligned}
$$

The integrand $x(f(x))^2(1 - f(x))$ is nonnegative, because $x \geq 0$, $(f(x))^2 \geq 0$ and $f(x) \leq 1$ a.e. on $[0,1]$. The integrand is also measurable, because f and x are measurable, and so is $(1 - f(x))$.

The integral of a nonnegative measurable function is 0 if and only if the integrand is 0 a.e. (Proposition 9, page 80 of Royden Fitzpatrick):

$$x(f(x))^2(1 - f(x)) = 0 \text{ a.e. on } [0, 1]$$

Since $x = 0$ only at 0, this means, $(f(x))^2(1 - f(x)) = 0$ a.e. on $(0, 1]$.
So either $f(x) = 0$ or $f(x) = 1$ a.e. on $(0, 1]$.
It remains to show that $m(\{x \in [0, 1] : f(x) = 1\}) > 0$.
If $m(\{x \in [0, 1] : f(x) = 1\}) = 0$, then $f(x) = 0$ a.e. on $[0, 1]$, which would mean $\int_0^1 xf(x)dx = 0$ (by the same proposition above: prop 9 from Royden Fitzpatrick). Since we are assuming that $\in_0^1 xf(x)dx$ is nonzero, this cannot be the case. Thus, $m(\{x \in [0, 1] : f(x) = 1\}) > 0$.

□