Analysis Prelim January 2014

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Problem 1

Let X be a metric space, $A \subset X$ a compact subset and $p \in X \setminus A$ a point of X not in A. Prove that there exist disjoint open sets O_1 and O_2 in X such that $A \subset O_1$ and $p \in O_2$.

Solution:
EVERY METRIC SPACE IS HAUSDORFF.
For every $x \in A$, there exist open sets U_x, V_p such that $x \in U_x, p \in V_p$ and $U_x \cap V_p = \varnothing$.
Let $\{U_x\}_{x \in A}$ be a collection of such open sets for every $x \in A$, and let $\{V_k\}_{k \in \mathcal{I}}$ be the collection of corresponding open sets containing p.
$\{U_x\}_{x \in A}$ forms an open cover of A, so since A is compact, there exists a finite subcover $\{U_i\}_{i=1}^N \subseteq \{U_x\}_{x \in A}$.
Let $\{V_i\}_{i=1}^N \subseteq \{V_k\}_{k \in \mathcal{I}}$ be the corresponding open sets containing p.
Then, $U := \bigcup_{i=1}^N U_i$ is an open set containing A, and $V := \bigcap_{i=1}^N V_i$ is an open set containing p. Furthermore, by construction we have $U \cap V = \varnothing$, as desired.

□

Problem 2

Let $f(x)$ be a continuous real-valued function on $[0, 1]$ which satisfies

$$\int_0^1 f(x)x^n dx = 0 \text{ for } n = 0, 1, 2, ...$$

Prove that $f(x)$ is identically 0.
Hint: You may find the (Stone-)Weierstrass theorem useful.

Solution:
Note that since $\int_0^1 f(x)x^n dx = 0$, this implies that $\int_0^1 f(x)p(x)dx = 0$ for any polynomial $p(x)$.
The polynomial functions are dense in the continuous functions, so for any $\epsilon > 0$ there exists a sequence of polynomials $\{p_n(x)\}$ which converge to $f(x)$. Consider the integral:

$$\begin{aligned}
\left| \int_0^1 (f(x))^2 dx \right| &= \int_0^1 |f(x)| \cdot (f(x) - p_n(x) + p_n(x))dx \\
&=\leq \int_0^1 |f(x)| \cdot |f(x) - p_n(x)|dx + \left| \int_0^1 f(x)p_n(x)dx \right| \\
&\text{Since } \int_0^1 f(x)p_n(x)dx = 0 \text{ as noted above:} \\
&= \int_0^1 |f(x)| \cdot |f(x) - p_n(x)|dx \\
&\text{Taking the limit as } n \to \infty,\ |f(x) - p_n(x)| \to 0, \text{ so:} \\
&= 0
\end{aligned}$$

This implies that $(f(x))^2 = 0$ a.e., since this is a positive valued function. That implies that $f(x) = 0$ a.e., as desired.

□

Problem 3

Let f, g be nonnegative, measurable functions on $[0, 1]$ such that

$$\int_0^1 f(x)dx = 2$$

$$\int_0^1 g(x)dx = 1$$

$$\int_0^1 f(x)^2 dx = 5$$

Let $E = \{x \in [0, 1] | f(x) \geq g(x)\}$. Show that $m(E) \geq 1/5$ (m is the Lebesgue measure).

Solutions:
On E, $f(x) \geq g(x)$, so on E^c, we have $f(x) < g(x)$. By monotincity of the integral:

$$\begin{aligned}
\int_{E^c} f(x)dx &\leq \int_{E^c} g(x)dx \\
\int_0^1 f(x)dx - \int_E f(x)dx &\leq \int_0^1 g(x)dx - \int_E g(x)dx \\
2 - \int_E f(x)dx &\leq 1 - \int_E g(x)dx \\
-\int_E f(x)dx &\leq -1 - \int_E g(x)dx \\
\int_E f(x)dx &\geq 1 + \int_E g(x)dx
\end{aligned}$$

Since g is nonnegative, this implies that $\int_E f(x)dx \geq 1$.
Note that $f \in L^2([0, 1])$, since $\|f\|_2^2 = 5$ is given. We will use this fact so that we can apply H"older's inequality:

$$\begin{aligned}
1 &\leq \int_E f(x)dx \\
&= \int_0^1 f(x)\chi_E dx \\
&\text{By Hölder's Inequality: :} \\
&\leq \|f\|_2 \|\chi_E\|_2 \\
&= \sqrt{5} \cdot \sqrt{\mu(E)}
\end{aligned}$$

Squaring both sides and dividing by 5, we get $\mu(E) \geq 1/5$, as desired.

□

Problem 4

Assume that $f : [0, 1] \to \mathbb{R}$ is an absolutely continuous function with $\int_0^1 f(x)dx = 0$. Prove for any $y \in [0, 1]$ that

$$\left| \int_0^1 (y - x)f'(x)dx \right| \leq \sup_{0 \leq x \leq 1} |f(x)|$$

Solution:

Since f is absolutely continuous, we know $f'(x)$ exists for all $x \in [0,1]$ and $f(x) = f(0) + \int_0^x f(t)dt$.

$$\left|\int_0^1 (y-x)f'(x)dx\right| = \left|y\int_0^1 f'(x)dx - \int_0^1 xf'(x)dx\right|$$

Integrating the second integral by parts:

$$= \left|y\int_0^1 f'(x)dx - \left(xf(x)\Big|_0^1 - \int_0^1 f(x)dx\right)\right|$$

$$= \left|y\int_0^1 f'(x)dx - f(1)\right|$$

Using the FTC to evaluate the integral:

$$= |yf(1) - yf(0) - f(1)|$$

$$= |f(1)(y-1) - yf(0)|$$

Factoring out a -1:

$$= |(1-y)f(1) + yf(0)|$$

If y is some point in $[0,1]$, then $(1-y)f(1) + yf(0)$ is a parametrization of the straight line connecting $f(0)$ and $f(1)$.

This means that $(1-y)f(1) + yf(0) \leq \max_{x \in \{0,1\}} f(x)$.

Certainly we have $|\max_{x\in\{0,1\}} f(x)| \leq \sup_{0 \leq x \leq 1} |f(x)|$, so the desired result follows.

□

Problem 5

Let $f \in L^3([-1,1])$. Show that

$$\int_{-1}^1 \frac{|f(x)|}{\sqrt{|x|}} dx < \infty$$

Solution:

Since $f \in L^3([-1,1])$, if we show that $\frac{1}{\sqrt{|x|}} \in L^{3/2}([-1,1])$, then we can apply Hölder's inequality.

We need to show its $L^{3/2}$-norm is finite:

$$\int_{-1}^1 \left(\frac{1}{\sqrt{|x|}}\right)^{3/2} dx = \int_{-1}^1 |x|^{-3/4} dx$$

$$= \int_{-1}^0 (-x)^{-3/4} dx + \int_0^1 x^{-3/4} dx$$

$$= 2\int_0^1 x^{-3/4} dx$$

$$= 2(4x^{1/4}\Big|_0^1)$$

$$= 8$$

$$< \infty$$

which shows $\frac{1}{\sqrt{|x|}} \in L^{3/2}([-1,1])$.

Now Hölder's theorem applies and we must have $|f(x) \cdot \frac{1}{\sqrt{|x|}}| \in L^1([-1,1])$ and

$$\int_{-1}^1 \frac{|f(x)|}{\sqrt{|x|}} dx \leq \|f\|_{L^3} \cdot \left\|\frac{1}{\sqrt{|x|}}\right\|_{L^{3/2}} < \infty$$

□

Problem 6

(a) Show that for $x > 0$ the limit $\lim\limits_{R\to\infty} \int_0^R \frac{\cos(t)}{x+t} dt$ exists.

(b) Define for $x > 0$

$$f(x) = \lim_{R\to\infty} \int_0^R \frac{\cos(t)}{x+t} dt$$

Show that $f(x)$ is continuous on $(0, \infty)$.

Solution:

(a) Is noticing that

$$\frac{\cos(t)}{x+t} = \int_0^\infty e^{-(x+t)y} \cos(t) dy$$

worth anything?
...Maybe not.
Look at the integral and using integration by parts:

$$\begin{aligned}
\int_0^R \frac{\cos(t)}{x+t} dt &= \frac{\sin(t)}{x+t}\Big|_0^R + \int_0^R \frac{\sin(t)}{(x+t)^2} dt \\
&= \frac{\sin(R)}{x+R} - \frac{\sin(0)}{x} + \int_0^R \frac{\sin(t)}{(x+t)^2} dt \\
&\text{Since } \sin(t), \sin(R) \le 1\text{:} \\
&\le \frac{1}{x+R} + \int_0^R \frac{1}{(x+t)^2} dt \\
&= \frac{1}{x+R} + \left(\frac{-1}{(x+t)}\Big|_0^R \right) \\
&= \frac{1}{x+R} - \frac{1}{x+R} + \frac{1}{x} \\
&= \frac{1}{x}
\end{aligned}$$

Taking the limit as $R \to \infty$:

$$\lim_{R\to\infty} \int_0^R \frac{\cos(t)}{x+t} dt = \frac{1}{x}$$

Since the value $1/x$ is finite, the limit necessarily exists.

(b) To show that $f(x)$ is a continuous function?
Assume wlog that $x \le y$. To show f is continuous at $x \in (0, \infty)$:

$$\begin{aligned}
|f(x) - f(y)| &= \lim_{R\to\infty} \left| \int_0^R \cos(t) \left(\frac{1}{x+t} - \frac{1}{y+t} \right) dt \right| \\
&= \lim_{R\to\infty} \left| \int_0^R \cos(t) \left(\frac{y-x}{(x+t)(y+t)} \right) dt \right| \\
&\text{By the change of variables } z = x + t\text{:} \\
&= |y-x| \lim_{R\to\infty} \int_x^R \frac{1}{z \cdot |y - x + z|} dz \\
&= |y-x| \cdot \frac{\log(y/x)}{y-x}
\end{aligned}$$

So if x, y are chosen such that $\log(y/x) < \epsilon$, then the result holds.

□