Analysis Prelim January 2013

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Problem 1

Let $f \in L^\infty([0,1])$, $f \neq 0$. Show that

$$\lim_{p\to\infty} \frac{\int_0^1 |f|^{p+1} dx}{\int_0^1 |f|^p dx}$$

exists and compute it.

Solution:
Note that $f \in L^\infty([0,1])$ implies that $f \in L^p([0,1])$ for all $p \in [1,\infty]$, because:

$$\begin{aligned}\int_0^1 |f(x)|^p dx &\leq \int_0^1 \|f\|_\infty^p \, dx \\ &= \|f\|_\infty^p \\ &< \infty, \text{ since } f \in L^\infty([0,1])\end{aligned}$$

Considering the quotient in question:

$$\begin{aligned}\frac{\int_0^1 |f|^{p+1} dx}{\int_0^1 |f|^p dx} &\leq \frac{\|f\|_\infty \int_0^1 |f|^p dx}{\int_0^1 |f|^p dx} \\ &= \|f\|_\infty\end{aligned}$$

Taking the limit:

$$\lim_{p\to\infty} \frac{\int_0^1 |f|^{p+1} dx}{\int_0^1 |f|^p dx} \leq \|f\|_\infty < \infty$$

So we see that the limit exists.
Claim: $\lim_{p\to\infty} \frac{\int_0^1 |f|^{p+1} dx}{\int_0^1 |f|^p dx} = \|f\|_\infty$
We need to show that the reverse inequality holds to prove the claim:

$$\text{NTS: } \lim_{p\to\infty} \frac{\int_0^1 |f|^{p+1} dx}{\int_0^1 |f|^p dx} \geq \|f\|_\infty$$

We will show this by first showing that $\|f\|_p \leq \|f\|_{p+1}$. First, note that the conjugate of $p+1$ is $(p+1)/p$:

$$\frac{1}{p+1} + \frac{p}{p+1} = 1$$

Using this relationship with Hölder's Inequality:

$$\begin{aligned}\|f\|_p^p &= \int_0^1 |f(x)|^p dx \\ &\leq \|1\|_{p+1} \cdot \||f|^p\|_{(p+1)/p} \\ &= 1 \cdot \left(\int_0^1 |f(x)|^{p+1} dx\right)^{p/(p+1)} \\ &= \||f|^p\|_{p+1} \\ &= \|f\|_{p+1}^p\end{aligned}$$

So $\|f\|_p^p \leq \|f\|_{p+1}^p$, and when we take the p-th root, we get $\|f\|_p \leq \|f\|_{p+1}$. Using this information:

$$\begin{aligned}\frac{\int_0^1 |f|^{p+1}}{\int_0^1 |f|^p} &= \frac{\|f\|_{p+1}^{p+1}}{\|f\|_p^p} \\ &\geq \frac{\|f\|_{p+1}^{p+1}}{\|f\|_{p+1}^p} \\ &= \|f\|_{p+1}\end{aligned}$$

Taking the limit as $p \to \infty$:

$$\lim_{p\to\infty} \frac{\int_0^1 |f|^{p+1}dx}{\int_0^1 |f|^p dx} \geq \lim_{p\to\infty} \|f\|_{p+1} = \|f\|_\infty$$

The last equality is due to the fact that the limit of the L^p-norms is the L^∞-norm.

□

Problem 2

Is it true that for any $f \in L^1([0,1])$ there exists $[a,b] \subset [0,1], a < b$, such that $f \in L^2([a,b])$?

Solution:
No, it is not true.
Consider $f(x) = \frac{1}{\sqrt{x}}$. We know that this is in $L^1([0,1])$, but not in $L^1([0,1])$. To extend this so that there are blow-ups inside every interval, let $\{q_n\}_{n=1}^\infty$ be an enumeration of the rationals in $[0,1]$. Then, define:

$$f(x) = \sum_{n=1}^\infty \frac{1}{2^n\sqrt{|x-q_n|}}$$

This function is in $L^1([0,1])$, because this series is absolutely convergent:

$$\begin{aligned}\sum_{n=1}^\infty \left|\frac{1}{2^n\sqrt{|x-q_n|}}\right| &\leq \sum_{n=1}^\infty \frac{1}{2^{n+1}}, \text{ because } |x-q_n| \leq |x| + |q_n|, \text{ and } x, q_n \in [0,1]. \\ &= \frac{1/4}{1-1/2} \\ &= \frac{1}{2}\end{aligned}$$

And the function is absolutely integrable over $[0,1]$:

$$\int_0^1 |f(x)|dx = \int_0^1 \sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{|x-q_n|}}dx$$

We justify switching the sum and the integral by Tonelli's theorem, because the integrand is nonnegative

$$= \sum_{n=1}^{\infty} \int_0^1 \frac{1}{2^n\sqrt{|x-q_n|}}dx$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \int_0^1 (|x-q_n|)^{-1/2}dx$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \left(2(|x-q_n|)^{1/2}\Big|_0^1 \right)$$
$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}(|1-q_n|^{\frac{1}{2}} - |q_n|^{\frac{1}{2}})$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}|1-q_n|^{1/2}$$
$$\leq \sum_{n=1}^{\infty} \frac{1}{2^{n-2}}$$

$< \infty$, since this is a geometric series with ratio $1/2 < 1$.

For any interval $[a,b] \in [0,1]$, we can find a rational $q_k \in (a,b)$, since the rationals are dense in the reals. At this point, $|f|^2$ will have a blow-up point, and thus f cannot be in $L^1([a,b])$.

□

Problem 3

Let $E \subset [0,1]$ denote the set of all numbers x that have some decimal expansion $x = 0.a_1a_2a_3...$ with $a_n \neq 2$ for all n. Show that E is a measurable set, and calculate its measure.

Solution:
If E denotes the set of numbers in $[0,1]$ without 2 in their decimal expansion, then E^c is the set of all numbers where 2 does appear in its decimal expansion.
Define A_n to be the set of all numbers where 2 appears in the nth decimal place, and the nth decimal place is the first place where 2 appears.
Looking at these sets as disjoint unions:

$$A_1 = [.2, .3)$$

$$A_2 = \left(\bigcup_{i=0}^{1}[.i2, .i3)\right) \bigcup \left(\bigcup_{i=3}^{9}[.i2, .i3)\right)$$

$$A_3 = \left(\bigcup_{i=0}^{1}\bigcup_{j=0}^{1}[.ij2, .ij3)\right) \bigcup \left(\bigcup_{i=0}^{1}\bigcup_{j=3}^{9}[.ij2, .ij3)\right) \bigcup \left(\bigcup_{i=3}^{9}\bigcup_{j=0}^{1}[.ij2, .ij3)\right) \bigcup \left(\bigcup_{i=3}^{9}\bigcup_{j=3}^{9}[.ij2, .ij3)\right)$$

Continuing this way, we see A_n is the disjoint union of 9^{n-1} intervals, each of size 10^-n.
Since these unions are disjoint, and E^c is the disjoint union of A_n's, we can use the countable additivity of

measure:

$$\begin{aligned}
m(E^c) &= m\left(\bigcup_{n=1}^{\infty} A_n\right) \\
&= \sum_{n=1}^{\infty} m(A_n) \\
&= \sum_{n=1}^{\infty} \frac{9^{n-1}}{10^n} \\
&= \frac{1}{9}\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n \\
&= \frac{1}{9}\frac{9/10}{1-9/10} \\
&= 1
\end{aligned}$$

Since $m(E^c) = 1$, by excision property of measure $m(E) = 0$.

□

Problem 4

Show that if $A_n \subset [0,1]$ and Lebesgue-measurable, with measure at least $c > 0$ for each $n \geq 1$, then the set of points which belong to infinitely many sets is measurable and its measure is at least c.

Solution:
The set of all points which belong to infinitely many of the A_n sets is:

$$\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k$$

Let $\bar{A}_n := \bigcup_{k\geq n} A_k$. Note that $\bar{A}_1 \supseteq \bar{A}_2 \supseteq \cdots \supseteq \bar{A}_n \supseteq \bar{A}_{n+1} \supseteq \cdots$.
We also have $\mu(\bar{A}_n) > c$ for all n, because $\bar{A}_n$ contains A_k for all $k \geq n$.
This allows us to express the set of all points which belong to infinitely many of the A_n sets as:

$$\bigcap_{n=1} \bar{A}_n$$

Taking the measure of the set of all points which belong to infinitely many of the A_n:

$$\mu\left(\bigcap_{n=1}^{\infty} \bigcup_{k\geq n} A_k\right) = \mu\left(\bigcap_{n=1} \bar{A}_n\right)$$

Since the A_n are nested:

$$\begin{aligned}
&= \lim_{n\to\infty} \mu(\bar{A}_n) \\
&\geq c
\end{aligned}$$

□

Problem 5

Construct Lebesgue-measurable real valued functions on $[a,b]$ so that they converge to zero pointwise but there is no null set N in $[a,b]$ such that the convergence is uniform outside of N. (i.e., Egoroff's Theorem is sharp.)

Solution:
The standard counterexample for Egoroff's is for how Egoroff's doesn't hold with the measure of the space is infinite. For example $f_n(x) = \chi_{[n,n+1]}$. This sequence of functions converges to 0 pointwise, but there's no set (size ϵ or less) that we can remove from $[0, \infty)$ to make the convergence uniform everywhere else.

In this case, we wish to show that the set we remove can't always be measure 0. It can be arbitrarily small, but not always null.
Consider the functions $\{f_n\}$ defined on $[a, b]$ via:

$$f_n(x) := \chi_{(a,a+\frac{b-a}{n}]} \text{ for } n = 1, 2, ...$$

$f_n \to 0$ pointwise:
$f_n(a) = 0$ for all n, and for any $x \in (a, b]$, there exists $n \in \mathbb{N}$ such that $x \notin (a, a + \frac{b-a}{n}]$, and then $f_n(x) = 0$.
There is no null set we can remove to make the convergence uniform:

$$f_n(x) \neq 0 \text{ for } x \in (a, a + \frac{b-a}{n}] \text{ and } m((a, a + \frac{b-a}{n}]) = \frac{b-a}{n} > 0$$

So we will not be able to remove a null set, even though we can choose n to make the measure of the set arbitrarily small. So Egoroff's theorem holds, but the sets removed will not be null sets.

□

Problem 6

Prove that for any function $f : \mathbb{R} \to \mathbb{R}$, the set of its continuity points is G_δ.

Solution:
A G_δ set is a countable intersection of open sets.
We define the oscillation of f on a subset S of its domain by:

$$\omega_f(S) := \text{diam}(f(S)) = \sup_{x,x' \in S} |f(x) - f(x')|$$

We define the oscillation of f at a point x of its domain by:

$$\omega_f(x) := \inf_{\epsilon > 0} \omega_f(B_\epsilon(x))$$

where $B_\epsilon(x)$ is the open ball of radius ϵ centered at x.
f is continuous at a point x if and only if $\omega_f(x) = 0$: This is because the differences in function value get arbitrarily small as the domain values get arbitrarily close, for a continuous function f.
The sets $\{x : \omega_f(x) < c\}$ are open, because if $\omega_f(x_0) < c$, then $\omega_f(x) < c$ in a sufficiently small neighborhood of x_0, because of how $\omega_f(x)$ is defined.
The set of continuity points of f can be defined:

$$\bigcap_{n=1}^{\infty} \{x : \omega_f(x) < \frac{1}{n}\}$$

this is a countable intersection of open sets, which is a G_δ set, as desired.

□