# Analysis Prelim August 2013

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## Problem 1

Prove the following "modified squeeze law":
Suppose we have real numbers $a_n, B_{m,n}$, and $c_m$ such that

$$0 \leq a_n \leq b_{m,n} + c_m$$

for all $m, n$ sufficiently large (say, greater than some fixed integer $K$). If

$$\lim_{m\to\infty} c_m = 0$$

and, for any fixed $m$,

$$\lim_{n\to\infty} b_{m,n} = 0$$

then $\lim_{n\to\infty} a_n = 0$.

*Solution:*
Fix $\epsilon > 0$
Since $\lim_{m\to\infty} c_m = 0$, there exists $M \in \mathbb{N}$ such that if $m \geq M$ then $|c_m - 0| < \epsilon/2$. Fixing this $M$, we get:

$$0 \leq a_n \leq b_{M,n} + c_M \Rightarrow 0 \leq a_n \leq b_{M,n} + \epsilon/2$$

Taking the limit as $n \to \infty$:

$$0 \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_{M,n} + \epsilon/2$$
$$0 \leq \lim_{n\to\infty} a_n \leq \epsilon/2 < \epsilon$$

Because $\lim_{n\to\infty} b_{M,n} = 0$ for any fixed $M$.
Since $\epsilon$ was chosen arbitrarily, $\lim_{n\to\infty} a_n = 0$.

□

## Problem 2

Prove or disprove the following:
Let $f, g : \mathbb{R} \to \mathbb{R}$, and suppose the composite function $f \circ g$ is continuous everywhere. If $\lim_{u\to b} f(u) = c$ and $\lim_{x\to a} g(x) = b$ (for $b, c \in \mathbb{R}$), then $\lim_{x\to a} f(g(x)) = c$.

*Solution:*
False. Consider the function $g(x) = 0$, and the function $f(x)$ defined:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then, $\lim_{x\to 0} g(x) = 0$, $\lim_{u\to 0} f(u) = 0$, but

$$\lim_{x\to 0} f(g(x)) = f(0) = 1$$

□

# Problem 3

The *convolution*, denoted $f * g$ of two functions $f, g : \mathbb{R} \to \mathbb{R}$ is defined by

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy$$

for any $x \in \mathbb{R}$ such that the integral on the right exists (in the Lebesgue sense).

(a) Show that, if $f, g \in L^2(\mathbb{R})$, then $f * g(x)$ exists for all $x$, that $f * g$ is bounded on $\mathbb{R}$, and that

$$\sup_{x \in \mathbb{R}} |f * g(x)| \leq \|f\|_2 \cdot \|g\|_2$$

(b) Show that, if $f, g \in L^1(\mathbb{R})$, then $f * g(x)$ exists for all $x$, that $f * g$ is bounded on $\mathbb{R}$, and that

$$\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$$

*Solution:*

(a)

$$\begin{aligned}
|f * g(x)| &= \left| \int_{\mathbb{R}} f(x-y)g(y)dy \right| \\
&\leq \int_{\mathbb{R}} |f(x-y)g(y)|dy \\
&\text{By Hölder's Inequality:} \\
&\leq \left( \int_{\mathbb{R}} |f(x-y)|^2 dy \right)^{1/2} \cdot \left( \int_{\mathbb{R}} |g(y)|^2 dy \right)^{1/2} \\
&\leq \left( \int_{-\infty}^{\infty} |f(x-y)|^2 dy \right)^{1/2} \cdot \|g\|_2 \\
&\text{Using the change of variables } s = x - y : \\
&= \left( - \int_{\infty}^{-\infty} |f(s)|^2 ds \right)^{1/2} \cdot \|g\|_2 \\
&= \left( \int_{-\infty}^{\infty} |f(s)|^2 ds \right)^{1/2} \cdot \|g\|_2 \\
&= \|f\|_2 \cdot \|g\|_2
\end{aligned}$$

Taking the supremum of both sides does not affect the right side, so we get

$$\sup_{x \in \mathbb{R}} |f * g(x)| \leq \|f\|_2 \cdot \|g\|_2$$

(b)

$$
\begin{aligned}
\|f * g\|_1 &= \int_{\mathbb{R}} |f * g(x)| dx \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x-y)g(y)| dy \right) dx \\
&\text{We will justify switching order of integrals by Fubini's:} \\
&= \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x-y)g(y)| dx \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \left( \int_{\mathbb{R}} |f(s)| ds \right) dy \\
&= \int_{\mathbb{R}} |g(y)| \cdot \|f\|_1 \, dy \\
&= \|f\|_1 \int_{\mathbb{R}} |g(y)| dy \\
&= \|f\|_1 \cdot \|g\|_1
\end{aligned}
$$

Since $f, g \in L^1(\mathbb{R})$, this is finite and Fubini's theorem does indeed justify switching the order of integration.

□

## Problem 4

The *Fourier transform*, denoted $\widehat{f}$, of a function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$\widehat{f}(s) = \int_{\mathbb{R}} f(x) e^{-2\pi i s x} dx$$

for any $s \in \mathbb{R}$ such that the integral on the right exists (in the Lebesgue sense).

(a) Show that, if $\fallingdotseq \in L^1(\mathbb{R})$, then $\widehat{f}(s)$ exists for all $s$, that $\widehat{f}$ is bounded and continuous and that

$$\sup_{s \in \mathbb{R}} |\widehat{f}(s)| \leq \|f\|_1$$

(b) Show that, if $f, g \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} \widehat{f}(u) g(u) du = \int_{\mathbb{R}} f(v) \widehat{g}(v) dv$$

*Solution:*

(a)

$$
\begin{aligned}
|\widehat{f}(s)| &= \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi i s x} dx \right| \\
&\leq \int_{-\infty}^{\infty} |f(x) e^{-2\pi i s x}| dx \\
&\text{And since } |e^{-2\pi i s x}| \leq 1 : \\
&\leq \int_{-\infty}^{\infty} |f(x)| dx \\
&= \|f\|_1
\end{aligned}
$$

Taking the sup on both sides does not affect the right side, so we obtain:

$$\sup_{s\in\mathbb{R}} |\widehat{f}(s)| \leq \|f\|_1$$

The Fourier transform is a linear operator on $L^1(\mathbb{R})$, so boundedness is the same as continuity.

(b) Using the fact that $\sup_{s\in\mathbb{R}} |\widehat{f}(s)| \leq \|f\|_1$ from part (a):

$$\begin{aligned}
\int_{\mathbb{R}} \widehat{f}(u)g(u)du &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)e^{-2\pi iux}dx\right) g(u)du \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)e^{-2\pi iux}g(u)dx\right) du \\
&\text{We wish to switch the order of integration.} \\
&\text{We can justify this switch later by Fubini's.} \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x)e^{-2\pi iux}g(u)du\right) dx \\
&= \int_{\mathbb{R}} f(x) \left(\int_{\mathbb{R}} e^{-2\pi iux}g(u)du\right) dx \\
&= \int_{\mathbb{R}} f(x)\widehat{g}(x)dx
\end{aligned}$$

□

## Problem 5

Give an example of a subset of $\mathbb{R}$ that is not a $G_\delta$ set. (Recall that a $G_\delta$ set in $\mathbb{R}$ is a countable intersection of open subsets of $\mathbb{R}$.) Can such a set be countable? If so, give an example ot show this. If not, explain why not.

*Solution:*
The set of rational numbers is not a countable intersection of open subsets of $\mathbb{R}$, so it is not a $G_\delta$ set. We will show this by contradiction.
Suppose $\mathbb{Q}$ is a countable intersection of open sets, say:

$$\mathbb{Q} = \bigcap_{n=1}^{\infty} \mathcal{O}_n$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, each set $\mathcal{O}_n$ must be dense in $\mathbb{R}$. The collection $\{\mathcal{O}_n\}_{n=1}^{\infty}$ is a countable collection of open, dense subsets of $\mathbb{R}$.
Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. The sets $\{\mathbb{R} \setminus \{r_k\}\}_{k=1}^{\infty}$ are dense in $\mathbb{R}$, and they are open sets since each set $\{r_n\}$ is closed. Thus, the collection $\{\mathbb{R} \setminus \{r_k\}\}_{k=1}^{\infty}$ is a countable collection of open, dense subsets of $\mathbb{R}$. The intersection of this collection of sets is the set of irrational numbers.
The union of two countable sets is countable, so $\{\mathbb{R} \setminus \{r_k\}\}_{k=1}^{\infty} \bigcup \{\mathcal{O}_{\|}{}_{k=1}^{\infty}\}$ is a countable collection of open, dense subsets of $\mathbb{R}$.
However, the set of irrational numbers is the complement of the set of rational numbers, so the intersection of this collection of sets is empty. This is a contradiction of the Baire Category Theorem, as it says that in a complete metric space, the intersection of a countable number of dense sets is necessarily dense as well.

□

## Problem 6

Prove or disprove the following:
Let $A$ be a measurable subset of $\mathbb{R}$. Let $I = A \cap [a, b]$, where $[a, b]$ is a compact interval in $\mathbb{R}$, and let $f : I \to \mathbb{R}$. Assume $f$ is continuous on $I$, in the sense that

$$\lim_{n\to\infty} x_n = x \Rightarrow \lim_{n\to\infty} f(x_n) = f(x)$$

for $x_n, x \in I$. Then, $f$ is bounded.

*Solution:*
**FALSE.** Let $I = (a, b)$ and take $f(x) = \frac{1}{x-a}$. $f$ is a continuous function on $I$, but $f$ is not bounded, because $f \to \infty$ as $x \to a$.

$\square$