Analysis Prelim January 2011

Sarah Arpin
University of Colorado Boulder
Mathematics Department
Sarah.Arpin@colorado.edu

Problem 1

Let $\{f_n\}$ be a sequence of measurable real-valued functions on $[0,1]$. Show that the set of x for which $\lim_{n\to\infty} f_n(x)$ exists is measurable.

Solution:
The lim sup and lim inf of sequences of measurable functions are measurable functions.
The set of $x \in [0,1]$ for which $\lim_{n\to\infty} f_n(x)$ exists is the set:

$$\{x \in [0,1] : \limsup_{n\to\infty} f_n(x) - \liminf_{n\to\infty} f_n(x) = 0\}$$

This set can also be expressed as an intersection:

$$\bigcap_{n=1}^{\infty} \{x \in [0,1] : \limsup_{n\to\infty} f_n(x) - \liminf_{n\to\infty} f_n(x) < 1/n\}$$

Since $\limsup - \liminf$ is measurable, this is an intersection of measurable sets, and thus is measurable itself.

□

Problem 2

Let $\{f_n\}$ be a sequence of measurable functions and suppose that

$$\sum_{n=1}^{\infty} m(\{x \in [0,1] : f_n(x) > 1\}) < \infty$$

where m is Lebesgue measure on $[0,1]$. Prove that $\limsup f_n(x) \leq 1$ for almost every $x \in [0,1]$.

Solution:
By the Borel-Cantelli Lemma, since the specified series converges, almost every $x \in [0,1]$ belongs to at most finitely many of the sets $\{x \in [0,1] : f_n(x) > 1\}$. There exists $N \in \mathbb{N}$ such that $m(\{x \in [0,1] : f_k(x) > 1\}) = 0$ for $k \geq N$.
Looking at the complement of this set, almost every x satisfies $f_k(x) \leq 1$ for $k \geq N$. So $\sup_{k\geq N} f_k(x) \geq 1$, and thus $\limsup_{k\geq N} \geq 1$.

□

Problem 3

(a) Let f be a real-valued Lebesgue measurable function defined on $[0,1]$. Give the definition of the essential supremum of f, $\|f\|_\infty$, and prove that if f and g are real-valued functions defined on $[0,1]$ whose essential supremums are finite, then $f+g$ is defined for almost all $x \in [0,1]$.

(b) Let $f : [0,1] \to \mathbb{R}$ be a Lebesgue measurable function with $\|f\|_\infty < \infty$. Prove that

$$\|f\|_\infty = \sup\left\{\left|\int_{[0,1]} f(x)g(x)dx\right| : g \in L^1[0,1], \|g\|_1 = 1\right\}$$

Solution:

(a) $f+g$ is defined by $f(x)+g(x)$.
Since $\|f\|_\infty$ and $\|g\|_\infty$ are both finite, that means that $f(x)$ and $g(x)$ are both finite almost everywhere. By possibly excising two sets of measure 0, this means that $f(x)+g(x)$ is finite a.e. Thus, $f+g$ is defined for almost all $x \in [0,1]$.

(b) By Hölder's Inequality, if $g \in L^1$ and $f \in L^\infty$, then $fg \in L^1$ and:

$$\int_{[0,1]} |f(x)g(x)|dx \leq \|f\|_\infty \|g\|_1$$

If we choose $g \in L^1$ such that $\|g\|_1 = 1$, then we will always have

$$\int_{[0,1]} |f(x)g(x)|dx \leq \|f\|_\infty$$

Thus, choosing $g \in L^1$ with $\|g\|_1 = 1$:

$$\left|\int_{[0,1]} f(x)g(x)dx\right| \leq \int_{[0,1]} |f(x)g(x)|dx \leq \|f\|_\infty$$

Taking the *sup*:

$$\left|\int_{[0,1]} f(x)g(x)dx\right| \leq \|f\|_\infty$$

To show the reverse inequality, choose an arbitrary $\epsilon > 0$ and let $a_\epsilon = \|f\|_\infty - \epsilon$.
Define the set:

$$E_{a_\epsilon} := \{x \in [0,1] : f(x) > a_\epsilon\}$$

By the definition of the sup-norm, $m(E_{a_\epsilon}) > 0$.
Define the function $g(x)$:

$$g(x) := \operatorname{sgn}(f)(x) \cdot \frac{\chi_{E_{a_\epsilon}}(x)}{m(E_{a_\epsilon})}$$

$g \in L^1[0,1]$ and $\|g\|_1 = 1$:

$$\|g\|_1 = \int_0^1 |g(x)|dx = \frac{1}{m(E_{a_\epsilon})} \int_{E_{a_\epsilon}} 1dx = 1$$

Futhermore:

$$
\begin{aligned}
\sup\left\{\left|\int_{[0,1]} f(x)g(x)dx\right| : g\in L^1[0,1], \|g\|_1 = 1\right\} &\geq \left|\int_0^1 f(x)g(x)dx\right| \\
&= \int_{E_{a_\epsilon}} \frac{f(x)}{m(E_{a_\epsilon})}dx \\
&> \int_{E_{a_\epsilon}} \frac{1}{m(E_{a_\epsilon})}dx \\
&= a_\epsilon \\
&= \|f\|_\infty - \epsilon
\end{aligned}
$$

Since ϵ can be chosen arbitrarily small, this shows

$$
\|f\|_\infty \leq \sup\left\{\left|\int_{[0,1]} f(x)g(x)dx\right| : g\in L^1[0,1], \|g\|_1 = 1\right\}
$$

We have shown that this inequality holds in both directions, so we must have:

$$
\|f\|_\infty = \sup\left\{\left|\int_{[0,1]} f(x)g(x)dx\right| : g\in L^1[0,1], \|g\|_1 = 1\right\}
$$

□

Problem 4

Suppose that $\{f_n\}_{n=1}^{\infty} \in L^\infty[a,b]$, where $-\infty < a < b < \infty$. Let $f \in L^1[a,b]$.

(a) Show that for all $n \geq 1$, $f_n \in L^1[a,b]$.

(b) If $f_n \to f$ in $L^1[a,b]$, and $\sup_{n\geq 1} \|f_n\|_\infty < \infty$, prove that $f \in L^\infty[a,b]$.

(c) Assuming part (b), prove that for all $p \in (1,\infty)$, $f_n \to f \in L^p[a,b]$.

Solution:

(a) Since $f_n \in L^\infty[a,b]$, $\|f\|_\infty < \infty$. Also, by definition of essential supremum, $|f_n| \leq \|f\|_\infty$ a.e. on $[a,b]$. Possibly excising a set of measure 0, consider the L^1-norm of an arbitrary f_n:

$$
\begin{aligned}
\|f_n\|_1 &= \int_a^b |f_n(x)|dx \\
&\leq \int_a^b \|f_n\|_\infty dx \\
&= \|f_n\|_\infty |b-a| \\
&< \infty
\end{aligned}
$$

Thus, since $\|f_n\|_\infty$ and $|b-a|$ are both finite, we see $\|f_n\|_1 < \infty$, and thus $f_n \in L^1[a.b]$.

(b) Convergence in L^1 implies convergence in measure, which implies that there is a subsequence which converges pointwise a.e. So, for a.e. $x \in [a,b]$:

$$
\lim_{k\to\infty} f_{n_k}(x) = f(x)
$$

$$
\begin{aligned}
\|f\|_\infty &= \sup_{x\in[a,b]} |f(x)| \\
&= \sup_{x\in[a,b]} \lim_{k\to\infty} |f_{n_k}(x)| \\
&\le \liminf_{k\to\infty} \sup_{x\in[a,b]} |f_{n_k}(x)| \\
&= \liminf_{k\to\infty} \|f_{n_k}\|_\infty \\
&\le \sup_{k\ge 1} \|f_{n_k}\|_\infty \\
&< \infty
\end{aligned}
$$

(c) Let $p \in (1,\infty)$. To show $f_n \to f$ in $L^p[a,b]$, we need to show $\|f - f_n\|_p \to 0$ as $n \to \infty$. Looking at the L^p-norm of the difference:

$$
\begin{aligned}
\|f - f_n\|_p &= \left(\int_a^b |f - f_n|^p\right)^{1/p} \\
&\text{By Hölder's Inequality:} \\
&\le \left(\||f - f_n|^p\|_1 \cdot \|1\|_\infty\right)^{1/p} \\
&= \|f - f_n\|_1 \\
&\text{Since } f_n \to f \text{ in } L^1\text{:} \\
&\to 0 \text{ as } n \to \infty
\end{aligned}
$$

□

Problem 5

(a) Prove that for every $x > 0$:

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt$$

(b) Prove that

$$\frac{\partial}{\partial x}\left[\frac{e^{-xt}(-t\sin(x) - \cos(x))}{t^2+1}\right] = e^{-xt}\sin(x)$$

(c) Using parts (a) and (b), prove that

$$\lim_{A\to\infty}\int_0^A \frac{\sin(x)}{x} dx = \frac{\pi}{2}$$

State any theorems that you are using in your proof.

Solution:

(a)

$$
\begin{aligned}
\int_0^\infty e^{-xt} dt &= \lim_{a\to\infty} \int_0^a e^{-xt} dt \\
&= \lim_{a\to\infty} \left(\frac{-1}{x} e^{-xt}\Big|_0^a\right) \\
&= \lim_{a\to\infty} \left(\frac{-1}{x \cdot e^{xa}} + \frac{1}{x \cdot e^{x\cdot 0}}\right) \\
&= \frac{1}{x}
\end{aligned}
$$

(b)

$$\begin{aligned}
\frac{\partial}{\partial x}\left[\frac{e^{-xt}(-t\sin(x)-\cos(x))}{t^2+1}\right] &= \frac{1}{t^2+1}\frac{\partial}{\partial x}(e^{-xt}(-t\sin(x)-\cos(x))) \\
&= \frac{1}{t^2+1}\left(-te^{-xt}(-t\sin(x)-\cos(x))+e^{-xt}(-t\cos(x)+\sin(x))\right) \\
&= \frac{e^{-xt}}{t^2+1}((t^2+1)\sin(x)) \\
&= e^{-xt}\sin(x)
\end{aligned}$$

(c) Begin by using the identity established in part (a):

$$\begin{aligned}
\lim_{A\to\infty}\int_0^A \frac{\sin(x)}{x}dx &= \lim_{A\to\infty}\int_0^A \sin(x)\left(\int_0^\infty e^{-xt}dt\right)dx \\
&= \lim_{A\to\infty}\int_0^A\int_0^\infty \sin(x)e^{-xt}dtdx
\end{aligned}$$

We will justify switching the order of integration by Fubini's Theorem:

$$= \int_0^\infty \lim_{A\to\infty}\int_0^A \sin(x)e^{-xt}dxdt$$

Using (b) to integrate:

$$\begin{aligned}
&= \int_0^\infty \lim_{A\to\infty}\left(\frac{e^{-xt}(-t\sin(x)-\cos(x))}{t^2+1}\Big|_0^A\right)dt \\
&= \int_0^\infty \lim_{A\to\infty}\left(\frac{e^{-tA}(-t\sin(A)-\cos(A)}{t^2+1}-\frac{-t\sin(0)-\cos(0)}{t^2+1}\right) \\
&= \int_0^\infty \frac{1}{t^2+1}dt \\
&= \frac{\pi}{2}
\end{aligned}$$

Since our result is finite, Fubini's Theorem justifies switching the order of integration in the third line.

□

Problem 6

Let f_n be a sequence of real valued C^1 functions on $[0,1]$ such that, for all n,

$$|f_n'(x)| \leq \frac{1}{\sqrt{x}} \text{ for } x>0$$

$$\int_0^1 f_n(x)dx = 0$$

Prove that the sequence has a subsequence that converges uniformly on $[0,1]$.

Solution:
By Arzela-Ascoli, we need to show that the family of functions $\{f_n\}$ is uniformly bounded and equicontinuous.

Investigating the equicontinuity first. Without loss of generality, assume $x > y$.

$$\begin{aligned}
|f_n(x) - f_n(y)| &= \left| f_n(0) + \int_0^x f_n'(t)dt - f_n(0) - \int_0^y f_n'(t)dt \right| \\
&= \left| \int_y^x f_n'(t)dt \right| \\
&\leq \int_y^x |f_n'(t)|dt \\
&\leq \int_y^x \frac{1}{\sqrt{t}}dt \\
&= 2\sqrt{x} - 2\sqrt{y}
\end{aligned}$$

For any $\epsilon > 0$, $2\sqrt{x} - 2\sqrt{y} < \epsilon$ for any $x, y \in [0,1]$ such that $|x - y| < \frac{1}{4}\epsilon^2$. So setting $\delta = \frac{1}{4}\epsilon^2$, we see that the family of functions is uniformly equicontinuous (the delta does not depend on the particular f_n function, nor does it depend on the choice of x.).
Now, show that $\{f_n\}$ is bounded.
We are given that $\int_0^1 f_n(x)dx = 0$ for all n. This implies, by the mean value theorem, that there exists c_n for every n such that $f_n(c_n) = 0$.
Now, express $f_n(x)$ as an integral:

$$f_n(x) = f_n(c_n) + \int_{c_n}^x f_n'(t)dt = \int_{c_n}^x f_n'(t)dt$$

We will use this integral expression to uniformly bound the f_n's:

$$\begin{aligned}
|f_n(x)| &= \left| \int_{c_n}^x f_n'(t)dt \right| \\
&\leq \int_{c_n}^x |f_n'(t)|dt \\
&\leq \int_{c_n}^x \frac{1}{\sqrt{t}}dt \\
&= 2\sqrt{x} - 2\sqrt{c_n} \\
&\text{Since } x, c_n \in [0,1]: \\
&\leq 2
\end{aligned}$$

Thus, the f_n are uniformly bounded by 2.
By the Arzela-Ascoli theorem, this means that the sequence $\{f_n\}$ must have a uniformly convergent subsequence.

$\square$