Analysis Prelim August 2011

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Problem 1

Let m be Lebesgue measure on the real line, and let B be a Borel subset with $m(B) < \infty$.

(a) Show that, for every $1 \leq p < \infty$, there exists a sequence of continuous functions φ_n with compact support such that $\varphi_n \to 1_B$ in L^p. Show details.

(b) Show that continuous functions with compact support are dense in $L^p(\mathbb{R})$.

Solution: Note that simple functions are dense in L^p. Show that $C_C(\mathbb{R})$ is dense in the simple functions.

(a) Since $m(B) < \infty$, for every $\epsilon > 0$ there exists a compact set $K \subset B$ such that $m(B \setminus K) < \epsilon$.
Let 1_K be the characteristic function of K. Then we have:

$$\begin{aligned}
\|1_B - 1_K\|_p &= \left(\int_{\mathbb{R}} (1_B - 1_K) dm \right)^{1/p} \\
&= \left(\int_{B \setminus K} 1 dm \right)^{1/p} \\
&= (m(B \setminus K))^{1/p} \\
&< \epsilon^{1/p}
\end{aligned}$$

If we define $\{1_n\}_{n=1}^{\infty}$ to be the sequence of characteristic functions of these compact subsets of B, where 1_n corresponds to the compact subset $K \subset B$ such that $m(B \setminus K) < 1/n$, then $1_n \to 1_B$ in L^p, from the work above. These functions all have compact support by definition.

(b) Recall that the simple functions are dense in $L^p(\mathbb{R})$. For any $f \in L^p(\mathbb{R})$, there exists a simple function $g(x) = \sum_{i=1}^{n} c_i 1_{A_i}$ which approximates f arbitrarily closely with respect to the L^p norm.
Since simple functions are finite linear combinations of indicator function, and we have already shown that indicator functions can be approximated arbitrarily closely by continuous functions of compact support. The continuous functions of compact support are dense in the simple functions, and the simple functions are dense in L^p, so the continuous functions of compact support are also dense in L^p.

$\square$

Problem 2

Let $\rho_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}$ where $x \in \mathbb{R}$ and $y > 0$. Given a bounded uniformly continuous function f on $\mathbb{R}$, let

$$u_f(x, y) = \int_{-\infty}^{\infty} \rho_y(x - z) f(z) dz$$

Show that

$$|u_f(x, y) - f(x)| \leq \omega_f(\delta) + 2 \|f\|_\infty (1 - \frac{2}{\pi} \arctan(\frac{\delta}{y}))$$

where

$$\omega_f(\delta) = \sup_{x, x' \in \mathbb{R}, |x - x'| < \delta} \{|f(x) - f(x')|\}$$

In particular, conclude from this that $u_f(x,y) \to f(x)$ uniformly as $y \to 0$ for such f.

Solution:
First, note that the integral of $\rho_y(x-z)$ with respect to z is 1:

$$\begin{aligned}
\int_{-\infty}^{\infty} \rho_y(x-z)dz &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{1}{(x-z)^2+y^2}dz \\
&= \frac{1}{\pi y}\int_{-\infty}^{\infty} \frac{1}{\left(\frac{x-z}{y}\right)^2+1}dz \\
&= \frac{1}{\pi y}\left(-y\arctan\left(\frac{x-z}{y}\right)\Big|_{-\infty}^{\infty}\right) \\
&= 1
\end{aligned}$$

Using this, we can bound the desired difference:

$$\begin{aligned}
|u_f(x-y)-f(x)| &= \left|\int_{-\infty}^{\infty}\rho_y(x-z)f(z)dz - f(x)\right| \\
&= \left|\int_{-\infty}^{\infty}\rho_y(x-z)f(z)dz - f(x)\int_{-\infty}^{\infty}\rho_y(x-z)dz\right| \\
&= \left|\int_{-\infty}^{\infty}\rho_y(x-z)(f(z)-f(x))dz\right| \\
&\text{Let } U_\delta := \{x' \in \mathbb{R} : |x-x'| < \delta\}: \\
&= \left|\int_{U_\delta}\rho_y(x-z)(f(z)-f(x))dz + \int_{U_\delta^c}\rho_y(x-z)(f(z)-f(x))dz\right| \\
&\leq \int_{U_\delta}\rho_y(x-z)|f(z)-f(x)|dz + \int_{U_\delta^c}\rho_y(x-z)|f(z)-f(x)|dz \\
&\text{On } U_\delta, \text{ we know } |f(z)-f(x)| \leq \omega_f(\delta): \\
&\leq \omega_f(\delta)\int_{U_\delta}\rho_y(x-z)dz + (2\|f\|_\infty)\int_{U_\delta^c}\rho_y(x-z)dz
\end{aligned} \tag{1}$$

We know:

$$\int_{U_\delta}\rho_y(x-z)dz + \int_{U_\delta^c}\rho_y(x-z)dz = \int_{-\infty}^{\infty}\rho_y(x-z)dz$$

We also know $\int_{U_\delta}\rho_y(x-z)dz \leq \int_{-\infty}^{\infty}\rho_y(x-z)dz$, so we can use that in the first integral (1), but to be more precise for what we need to show in the second integral, let's do the full calculation:

$$\begin{aligned}
\int_{U_\delta}\rho_y(x-z)dz &= \int_{x-\delta}^{x+\delta}\rho_y(x-z)dz \\
&\text{Using the anti-derivative work we've already established:} \\
&= \frac{-1}{\pi}\left(\arctan\left(\frac{x-z}{y}\right)\Big|_{x-\delta}^{x+\delta}\right) \\
&= \frac{-1}{\pi}\left(\arctan\left(\frac{-\delta}{y}\right) - \arctan\left(\frac{\delta}{y}\right)\right) \\
&= \frac{2}{\pi}\arctan\left(\frac{\delta}{y}\right)
\end{aligned}$$

This tells us $\int_{U_\delta^c}\rho_y(x-z)dz = 1 - \frac{2}{\pi}\arctan\left(\frac{\delta}{y}\right)$. Using this substitution in the second integral from (1):

$$|u_f(x-y)-f(x)| \leq \omega_f(\delta) + 2\|f\|_\infty\left(1-\frac{2}{\pi}\arctan\left(\frac{\delta}{y}\right)\right)$$

Which is the desired inequality.
We can conclude that $u_f(x,y) \to f(x)$ uniformly as $y \to 0$, because as $y \to 0$, $(1 - \frac{2}{\pi}\arctan(\delta/y)) \to 0$, so we can choose a y to make this term of the above expression arbitrarily small (say less than $\epsilon/2$). Also, we can choose δ to make $\omega_f(\delta) < \epsilon/2$, since f is uniformly continuous. Neither of these choices depend don x, so $u_f(x,y) \to f(x)$ uniformly as $y \to 0$.

$\square$

Problem 3

Let μ be a finite measure on $(0,1)$ such that μ and Lebesgue measure are mutually singular. Show that, for every $\epsilon \in (0, \mu(0,1))$, there exists a finite collection of disjoint open intervals (x_k, y_k) such that $\sum |y_k - x_k| < \epsilon$ and $\sum \mu(x_k, y_k) \geq \mu(0,1) - \epsilon$.

Solution:
Let m denote Lebesgue measure.
Let $\mu[0,1] = M > 0$. *We are extending μ to [0,1] so that we can use compactness.*
There exists $B \subset [0,1]$ such that $m(B) = 0$ and $\mu(B^c) = 0$. This is the definition of mutually singular.
There exists $U \supset B$ such that $m(U) < \epsilon$ (we can approximate sets on the outside by open sets with up to measure ϵ accuracy).
Since U is open, U is the union of at most countably many disjoint open intervals:

$$U = \bigcup_{k=1}^{\infty} (x_k, y_k)$$

We can use the countable additivity of measure to see:

$$\mu(U) = \mu\left(\bigcup_{k=1}^{\infty} (x_k, y_k)\right) = \sum_{k=1}^{\infty} \mu(x_k, y_k) \tag{2}$$

Also, using the additivity of measure and the fact that $U \supset B$ and $\mu(B^c) = 0$:

$$\mu(U) = \mu(U \cap B^c) + \mu(U \cap B) = \mu(B) = \mu([0,1] \setminus B^c) = M \tag{3}$$

Combining (2) and (3), we see:

$$\sum_{k=1}^{\infty} \mu(x_k, y_k) = M$$

Since this sum converges, for any $\epsilon > 0$, there exists N such that

$$\sum_{k=N+1}^{\infty} \mu(x_k, y_k) < \epsilon$$

and thus

$$\sum_{k=1}^{N} \mu(x_k, y_k) \geq M - \epsilon$$

Furthermore, since $\{(x_k, y_k)\}_{k=1}^{n} \subseteq U$, we have

$$\sum_{k=1}^{N} |y_k - x_k| \leq \sum_{k=1}^{\infty} |y_k - x_k| = m(U) < \epsilon$$

$\square$

Problem 4

Let $(\mathbb{R}, d_1)$ be the metric space which is the real line $\mathbb{R}$ with the usual complete Euclidean metric $d_1(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. If d_2 is another metric on $\mathbb{R}$ such that $(\mathbb{R}, d_2)$ a metric space with the same topology as $(\mathbb{R}, d_1)$, can we conclude that $(\mathbb{R}, d_2)$ is a complete metric space? Justify your answer.
Hint: Consider the function $\varphi(x) = \frac{x}{1+|x|}$ and $d_2(x, y) = |\varphi(x) - \varphi(y)|$.

Solution:
Being a complete metric space is not a topological invariant, so no.
The function $d_2(x, y) := |\varphi(x) - \varphi(y)|$ is a metric.

- Nonnegative (by definition) and Zero if and only if $x = y$:
- Symmetric
- Triangle Inequality

d_1 and d_2 generate the same topology on $\mathbb{R}$, because one $d_1(x, y) \to 0$ if and only if $d_2(x, y) \to 0$.
To see that $d_1(x, y) \to 0$ implies $d_2(x, y) \to 0$:

$$\begin{aligned}
d_2(x, y) &= \left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| \\
&= \left| \frac{x(1+|y|) - y(1+|x|)}{(1+|x|)(1+|y|)} \right| \\
&\text{The denominator is always } \geq 1 \text{, so:} \\
&\leq |x(1+|y|) - y(1+|x|)| \\
&= |x + x|y| - y + |x|y| \\
&= |x + x|y| - y + |x|y + y|y| - y|y|| \\
&= |x(1+|y|) - y(1+|y|) + y(|y| - |x|) \\
&\text{Since } |y| - |x| \leq |y - x| = |x - y| \text{:} \\
&\leq |(x-y)(1+|y|) + y(|x-y|) \\
&\leq |x - y|(1 + 2|y|)
\end{aligned}$$

To show $d_2(x, y) \to 0$ implies $d_1(x, y) \to 0$, we will break into three cases:

- If $x, y \geq 0$ and $x, y \leq M$, for some $M > 0$:

$$\begin{aligned}
\left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| &= \left| \frac{x}{1+x} - \frac{y}{1+y} \right| \\
&= \left| \frac{x(1+y) - y(1+x)}{(1+x)(1+y)} \right| \\
&\geq \left| \frac{x-y}{(1+M)^2} \right| \\
&= \frac{|x-y|}{(1+M)^2}
\end{aligned}$$

- If $x, y < 0$ and $x, y \geq -M$ for some $M > 0$:

$$\begin{aligned}
\left| \frac{x}{1+|x|} - \frac{y}{1+|y|} \right| &= \left| \frac{x}{1-x} - \frac{y}{1-y} \right| \\
&= \left| \frac{x(1-y) - y(1-x)}{(1-x)(1-y)} \right| \\
&\geq \left| \frac{x-y}{(1+M)^2} \right| \\
&= \frac{|x-y|}{(1+M)^2}
\end{aligned}$$

- If x, y are opposite signs, then for $d_2(x,y)$ to be small, both x and y have to be close to 0. Without loss of generality, suppose $|x|, |y| < 1$:

$$\begin{aligned}\left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right| &= \left|\frac{x(1+|y|-y-y|x|)}{(1+|x|)(1+|y|)}\right| \\ &= \left|\frac{x-y-2|xy|}{(1+|x|)(1+|y|)}\right| \\ &\geq \frac{|x-y|}{4}\end{aligned}$$

Which shows that convergence with respect to d_2 implies convergence in d_1.
This means that the topologies defined by the two metrics are the same.
However, this does *not* mean that $(\mathbb{R}, d_2)$ is a complete metric space.
In $(\mathbb{R}, d_2)$, the sequence $\{n\}_{n=1}^{\infty}$ is Cauchy, but it does not have a convergent subsequence.
First, to see that this sequence is Cauchy in $(\mathbb{R}, d_2)$. We may assume $n > 1$:

$$\begin{aligned}\left|\frac{n}{1+|n|} - \frac{n+1}{1+|n+1|}\right| &= \left|\frac{n}{1+n} - \frac{n+1}{n+2}\right| \\ &= \left|\frac{n(n+2)-(n+1)^2}{(n+1)(n+2)}\right| \\ &= \left|\frac{-1}{(n+1)(n+2)}\right|\end{aligned}$$

This expression will be less than ϵ iff $\frac{1}{(n+1)(n+2)} < \epsilon$, which is true iff $\frac{1}{\epsilon} < (n+1)(n+2) < (n+2)^2$. So as long as $n > \frac{1}{\sqrt{\epsilon}} - 2$, the expression will be less than ϵ. This means we have a Cauchy sequence.
However, it cannot converge. Remember that d_2 convergence would imply d_1 convergence, which is not possible: There is no number c such that $\lim\limits_{n\to\infty} |n-c| \to 0$.

$\square$

Problem 5

Let $L^2(-\pi, \pi)$ be the space of (complex-valued) absolutely square integrable functions on $(-\pi, \pi)$ with respect to Lebesgue measure. Denote

$$E_n(x) = e^{inx} \ ; n = 1, \pm 1, \pm 2, ...; \ -\pi \leq x \leq \pi$$

and

$$F_n(x) = E_{-n}(x) + nE_n(x) \ ; n = 1, 2, 3, ... \ ; \ -\pi \leq x \leq \pi$$

Let X_1 be the smallest closed subspace of $L^2(-\pi,\pi)$ that contains $E_0, E_1, ...$ Let X_2 be the smallest closed subspace of $L^2(-\pi,\pi)$ that contains $F_1, F_2, ...$

(a) Is $X_1 + X_2$, that is the linear span of X_1 and X_2, dense in $L^2(-\pi,\pi)$? Recall that if A and B are linear subspaces then $A + B = \{a + b : a \in A, b \in B\}$.

(b) Is $X_1 + X_2$ closed? Hint: Consider $\sum\limits_{n=1}^{\infty} n^{-1}E_n$.

Solution:

(a) Yes: $X_1 + X_2$ contains the $E_0, E_1, ...$ as well as $E_{-1} = F_1 - E_1, E_{-2} = F_2 - 2E_2, ...$ The $\{E_n\}_{n=-\infty}^{\infty}$ form an ON basis for $L^2(-\pi,\pi)$, so this is indeed a dense subspace.

(b) No. As per the hint (w/ typo...), we ask ourselves

$$\text{Is } \sum_{n=1}^{\infty} n^{-1}E_n \text{ in } X_1 + X_2?$$

If so, then there exist coefficients a_j, b_k such that

$$\sum_{n=1}^{\infty} n^{-1}E_{-n} = \underbrace{\sum_{j=0}^{\infty} a_j E_j}_{\in X_1} + \underbrace{\sum_{k=1}^{\infty} b_k(E_{-k} + kE_k)}_{\in X_2}$$

Using the inner product, we can figure out what the a_j and b_k coefficients would have to be in such an expansion.
Note $\langle E_k, E_j\rangle = 0$ if $k \neq j$ and $= 2\pi$ if $k = j$.

$$\left\langle \sum_{n=1}^{\infty} n^{-1}E_{-n}, E_0 \right\rangle = \left\langle \sum_{j=0}^{\infty} a_j E_j + \sum_{k=1}^{\infty} b_k(E_{-k} + kE_k), E_0 \right\rangle$$
$$0 = 2\pi a_0$$

So we know $a_0 = 0$. Now, suppose $k \geq 1$:

$$\left\langle \sum_{n=1}^{\infty} n^{-1}E_{-n}, E_k \right\rangle = \left\langle \sum_{j=0}^{\infty} a_j E_j + \sum_{k=1}^{\infty} b_k(E_{-k} + kE_k), E_k \right\rangle$$
$$0 = 2\pi a_k + 2\pi b_k k$$

And taking the inner product with E_{-k} gives another equation:

$$\left\langle \sum_{n=1}^{\infty} n^{-1}E_{-n}, E_{-k} \right\rangle = \left\langle \sum_{j=0}^{\infty} a_j E_j + \sum_{k=1}^{\infty} b_k(E_{-k} + kE_k), E_{-k} \right\rangle$$
$$2\pi k^{-1} = 2\pi b_k$$

So $b_k = k^{-1}$ for all $k \geq 1$, $a_j = -1$ for all $j \geq 1$, and $a_0 = 0$.
Putting this information back into our original expression for $\sum_{n=1}^{\infty} n^{-1}E_{-n}$ as an element of $X_1 + X_2$:

$$\sum_{n=1}^{\infty} n^{-1}E_{-n} = -\sum_{j=1}^{\infty} E_j + \sum_{k=1}^{\infty} \frac{1}{k}(E_{-k} + kE_k)$$

But the series $-\sum_{j=1}^{\infty} E_j$ does not converge, so we arrive at a contradiction.

$\square$

Problem 6

For a function $f \in L^p([0,\infty))$, $1 < p < \infty$, set

$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

(a) Assuming f is continuous nonnegative vanishing outside of a finite interval, show that

$$\int_0^{\infty} F^p(t)dt = \frac{p}{p-1}\int_0^{\infty} F^{p-1}(t)f(t)dt$$

Hint: Consider an equation relating F, f, and F'.

(b) For f as in part (a), establish Hardy's inequality:

$$\|F\|_p \leq \frac{p}{p-1}\,\|f\|_p$$

Hint: Try Hölder's inequality.

(c) Extend Hardy's inequality to all $f \in L^p([0,\infty))$.

Solution:

(a) Suppose f is continuous, nonnegative and vanishes outside of $[c,d]$, for $-\infty < c < d < \infty$.
To find the equation in the hint, begin with:

$$xF(x) = \int_0^x f(t)dt$$

and take the derivative on both sides with respect to x.

$$xF'(x) + F(x) = f(x) \Rightarrow xF'(x) = f(x) - F(x)$$

Using this after integrating by parts:

$$\int_0^\infty F^p(x)dx = xF^p(x)\Big|_0^\infty - p\int_0^\infty F^{p-1}(x)\cdot x\cdot F'(x)dx$$

Using the identity we noted above:

$$\int_0^\infty F^p(x)dx = \lim_{a\to\infty}\frac{1}{a^{p-1}}\left(\int_0^a f(x)dx\right)^p - \lim_{b\to 0}\frac{1}{b^{p-1}}\left(\int_0^b f(x)dx\right)^p - p\int_0^\infty F^{p-1}(x)f(x)dx + p\int_0^\infty F^{p-1}(x)F($$

$$(1-p)\int_0^\infty F^p(x)dx = -p\int_0^\infty F^{p-1}(x)f(x)dx$$

$$\int_0^\infty F^p(x)dx = \frac{p}{p-1}\int_0^\infty F^{p-1}(x)f(x)dx$$

(b) Beginning with the equation proved in part (a):

$$\|F\|_p^p = \frac{p}{p-1}\int_0^\infty F^{p-1}(t)f(t)dt$$

Using Hölder's Inequality and the fact that $p/(p-1)$ is the conjugate of p:

$$\leq \frac{p}{p-1}\left\|F^{p-1}\right\|_{p/(p-1)}\cdot\|f\|_p$$

$$= \frac{p}{p-1}\left(\int_0^\infty |F|^p\right)^{(p-1)/p}\cdot\|f\|_p$$

$$= \frac{p}{p-1}\|F\|_p^{p-1}\cdot\|f\|_p$$

Dividing both sides by $\|F\|_p^{p-1}$:

$$\|F\|_p \leq \frac{p}{p-1}\cdot\|f\|_p$$

(c) Continuous functions of compact support are dense in L^p, so once we extend the inequality to linear combinations of nonnegative continuous functions of compact support we are done.
Let g, f be continuous nonnegative functions of compact support, and let $c \in \mathbb{R}$.
Let $G(x) := \frac{1}{x}\int_0^x (f(t) + cg(t))dt$.
Using the same technique from part (a), we see:

$$xG'(x) = (f(x) + cg(x)) - G(x)$$

Starting again with integration by parts:

$$\int_0^\infty G^p(x)dx = xG^p(x)\Big|_0^\infty - p\int_0^\infty xG'(x)G^{p-1}(x)dx$$
$$\int_0^\infty G^p(x)dx = -p\int_0^\infty (f(x)+cg(x))G^{p-1}(x)dx + p\int_0^\infty G^p(x)dx$$

Combining like terms:

$$(1-p)\int_0^\infty G^p(x)dx = -p\int_0^\infty (f(x)+cg(x))G^{p-1}(x)dx$$
$$\int_0^\infty G^p(x)dx = \frac{p}{p-1}\int_0^\infty (f(x)+cg(x))G^{p-1}(x)dx$$

Now, we will use this identity together with the steps of part (b) to show the desired inequality:

$$\|G\|_p^p = \frac{p}{p-1}\int_0^\infty G^{p-1}(t)(f(t)+cg(t))dt$$

By Hölder's Inequality:

$$\|G\|_p^p \leq \frac{p}{p-1}(\left\|G^{p-1}\right\|_{p/(p-1)} \cdot \|f+cg\|_p)$$
$$\|G\|_p^p \leq \frac{p}{p-1}(\|G\|_p^{p-1} \cdot \|f+cg\|_p)$$
$$\|G\|_p \leq \frac{p}{p-1}\|f+cg\|_p$$

So we have shown Hardy's Inequality for continuous functions of compact support, because we have shown it for linear combinations of nonnegative continuous functions of compact support. The rest follows using the fact that continuous functions of compact support are dense in L^p with respect to the L^p norm.

□