Algebra Prelim: August 2015

1

Show that if the conjugacy classes of a finite group G have size at most 4, then G is solvable.

A group G is **solvable** if there is a chain of subgroups

$$1 = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft ... \triangleleft G_s = G$$

such that G_{i+1}/G_i is abelian for $i = 0, 1, ..., s-1$.

2

Show that if F is a nontrivial free group, then F has a proper subgroup of finite index.

Let F be a free group of rank n.
We will show that, for each $m \geq n$, F contains a free group of rank m as a finite-index subgroup.

3

Show that if R is a PID and S is an integral domain containing no subfield, then any homomorphism $\varphi : R \to S$ is injective.

Proof:
Need to show: